Graph Theory

Part Two

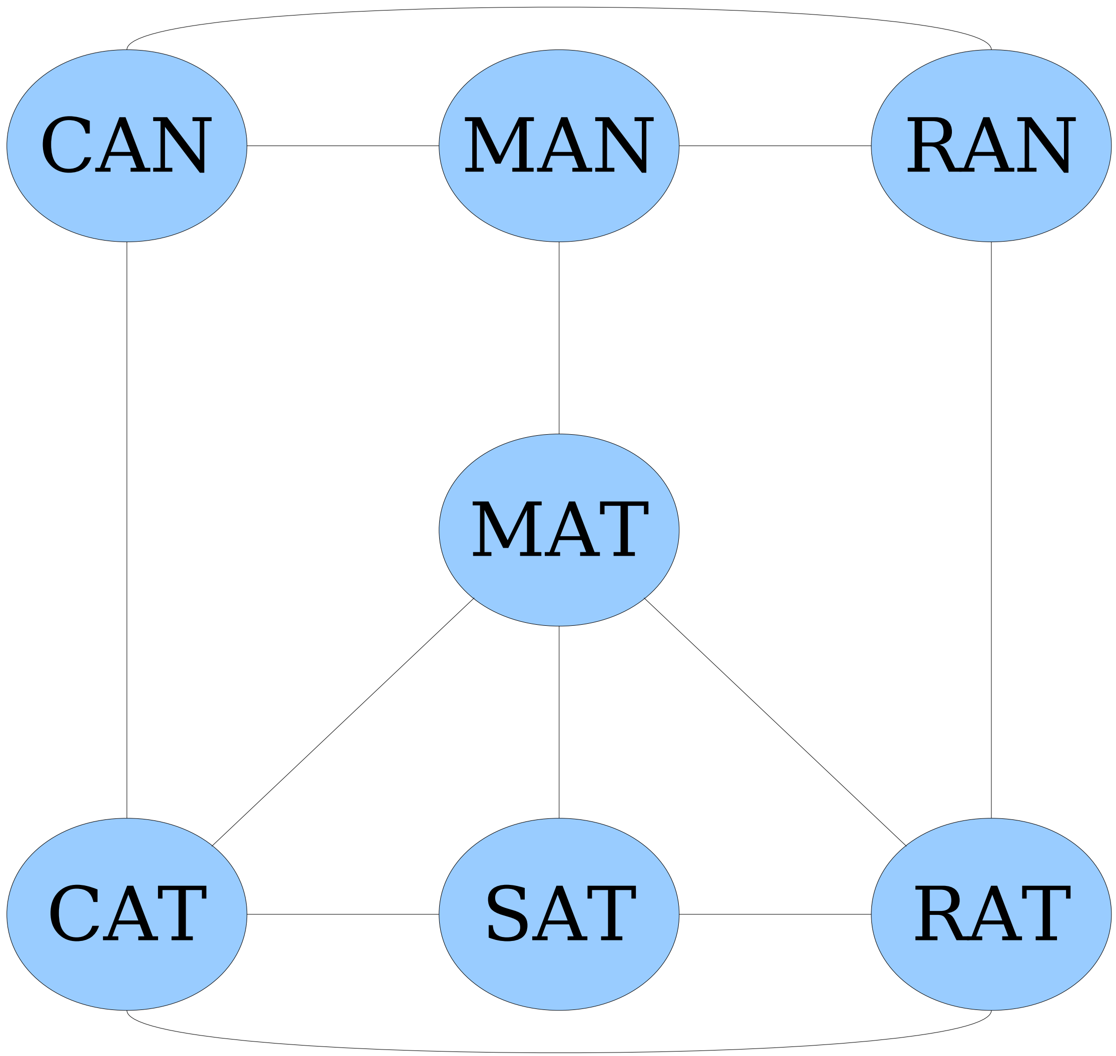
Outline for Today

- ***Walks, Paths, and Reachability***
  - Walking around a graph.
- ***Application: Local Area Networks***
  - Graphs meet computer networking.
- ***Trees***
  - A fundamental class of graphs.

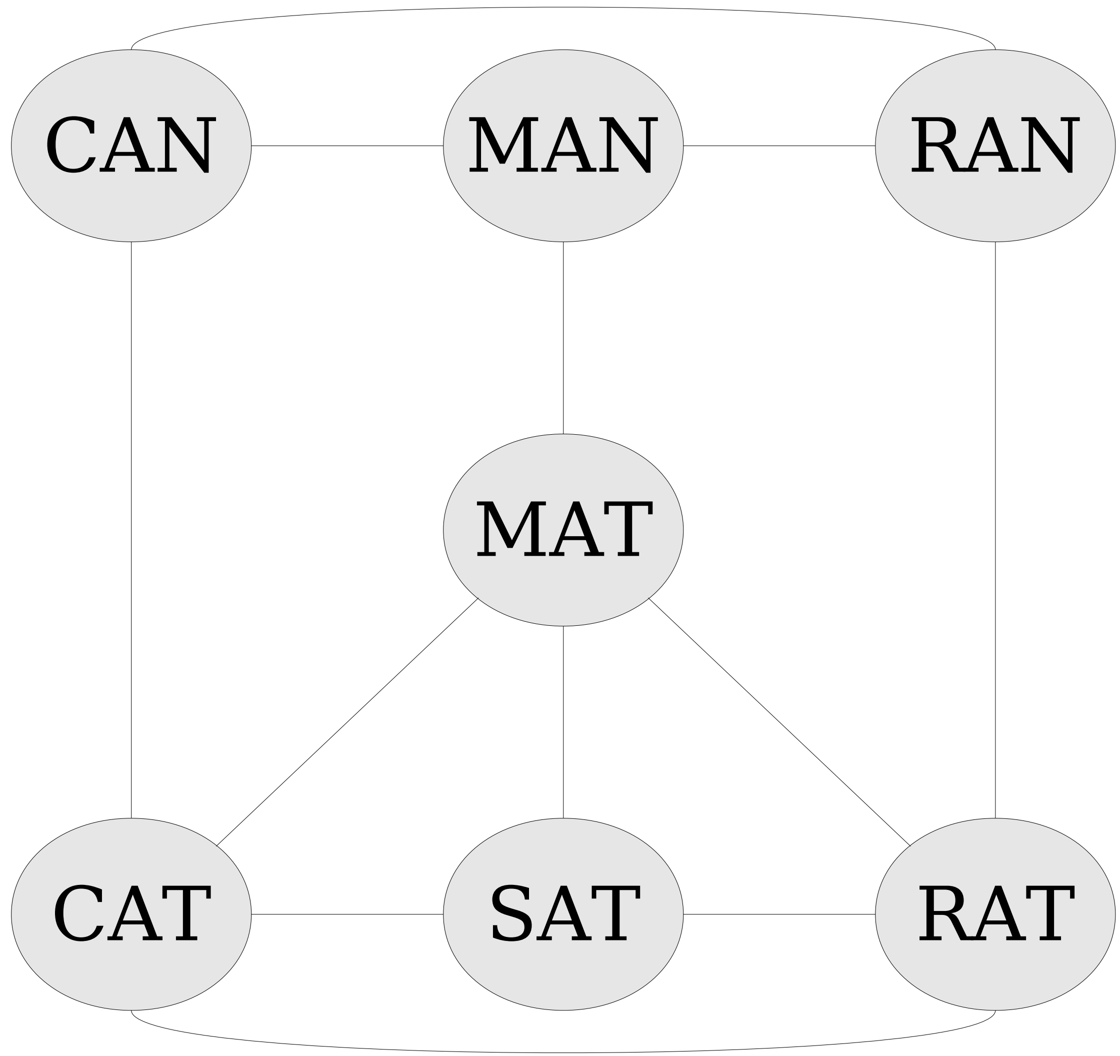
Recap from Last Time

Graphs and Digraphs

- A ***graph*** is a pair $G = (V, E)$ of a set of nodes V and set of edges E.
  - Nodes can be anything.
  - Edges are ***unordered pairs*** of nodes. If $\{u, v\} \in E$, then there's an edge from u to v.
- A ***digraph*** is a pair $G = (V, E)$ of a set of nodes V and set of directed edges E.
  - Each edge is represented as the ordered pair (u, v) indicating an edge from u to v.



Two nodes in an undirected graph are called ***adjacent*** if there is an edge between them.



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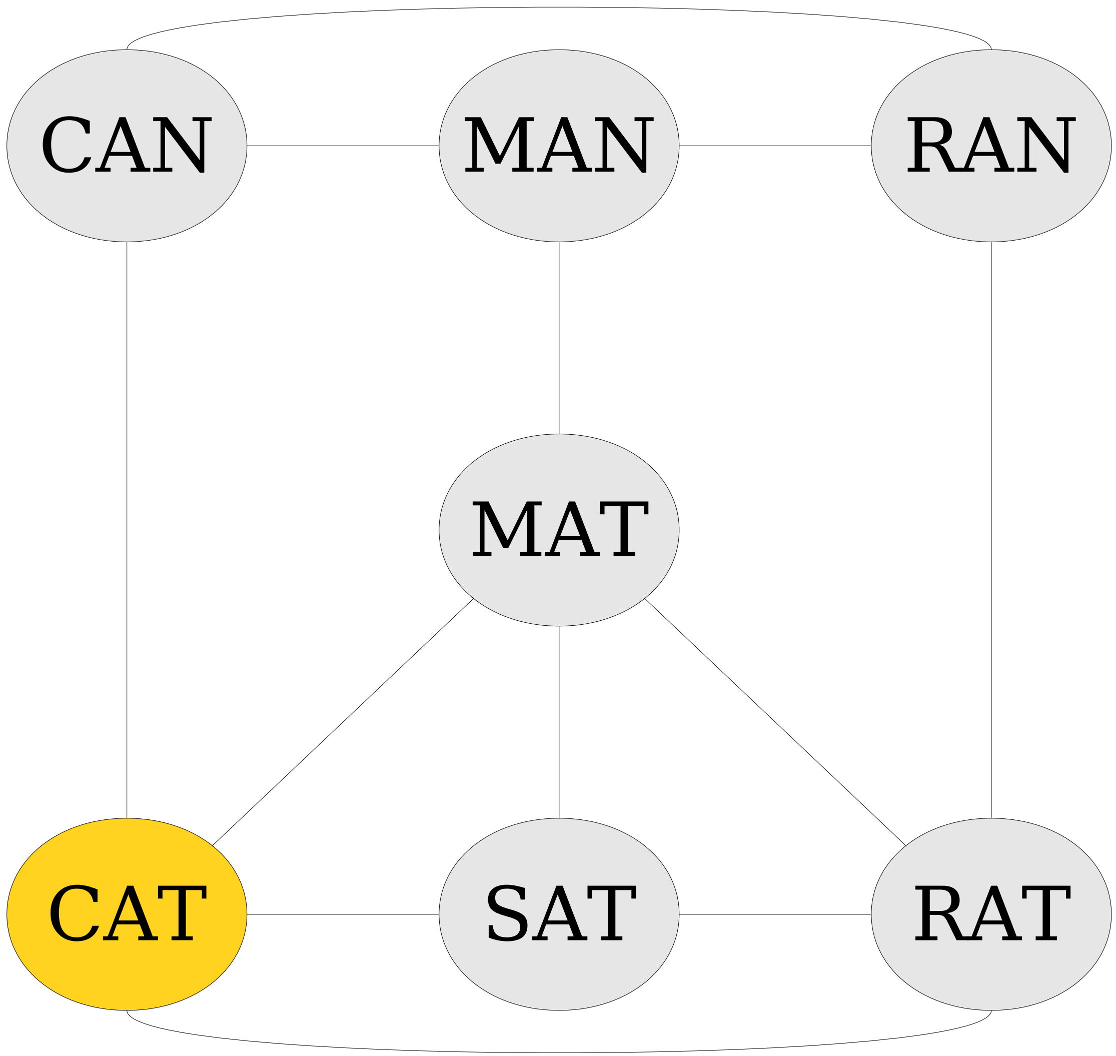

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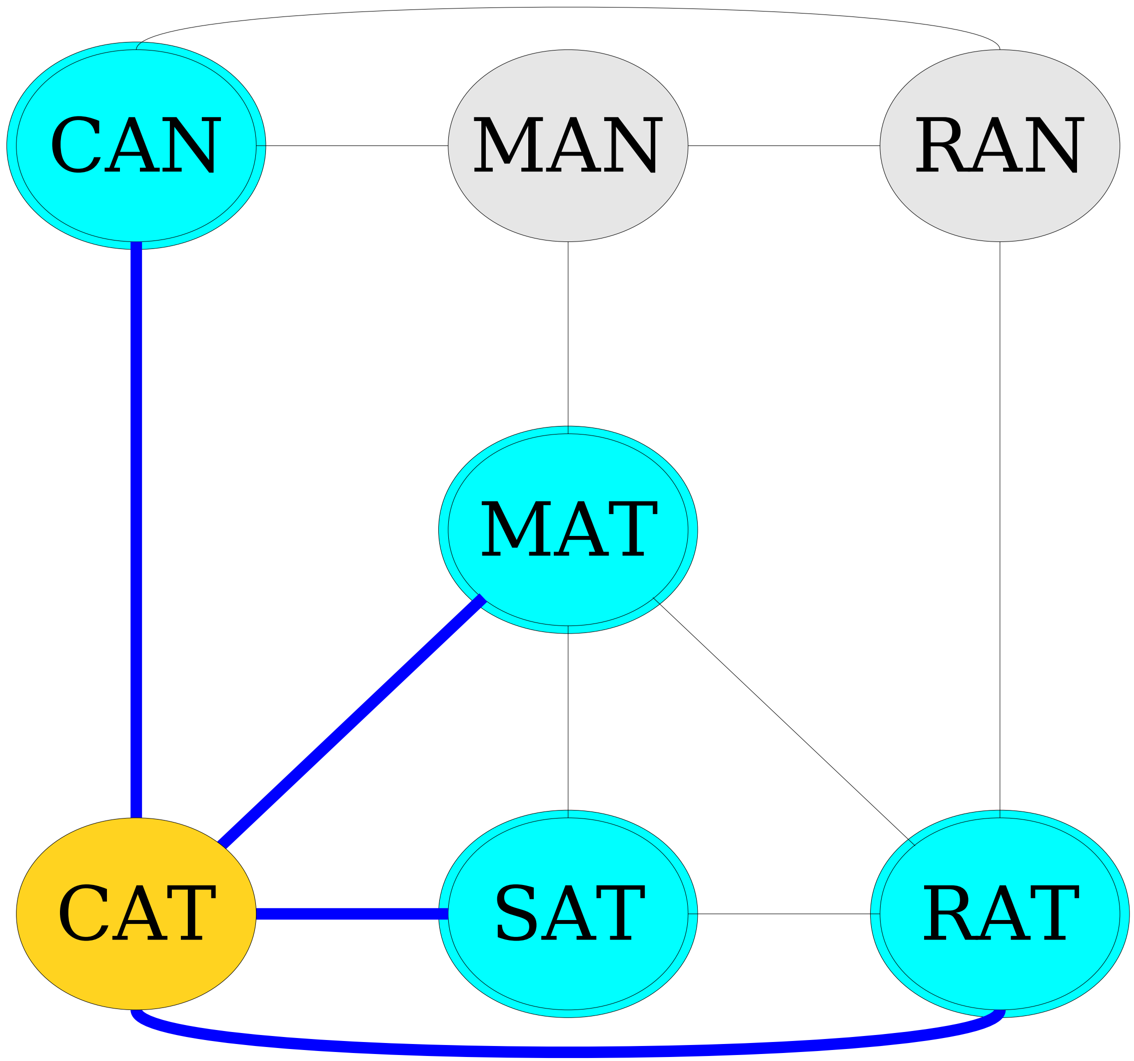

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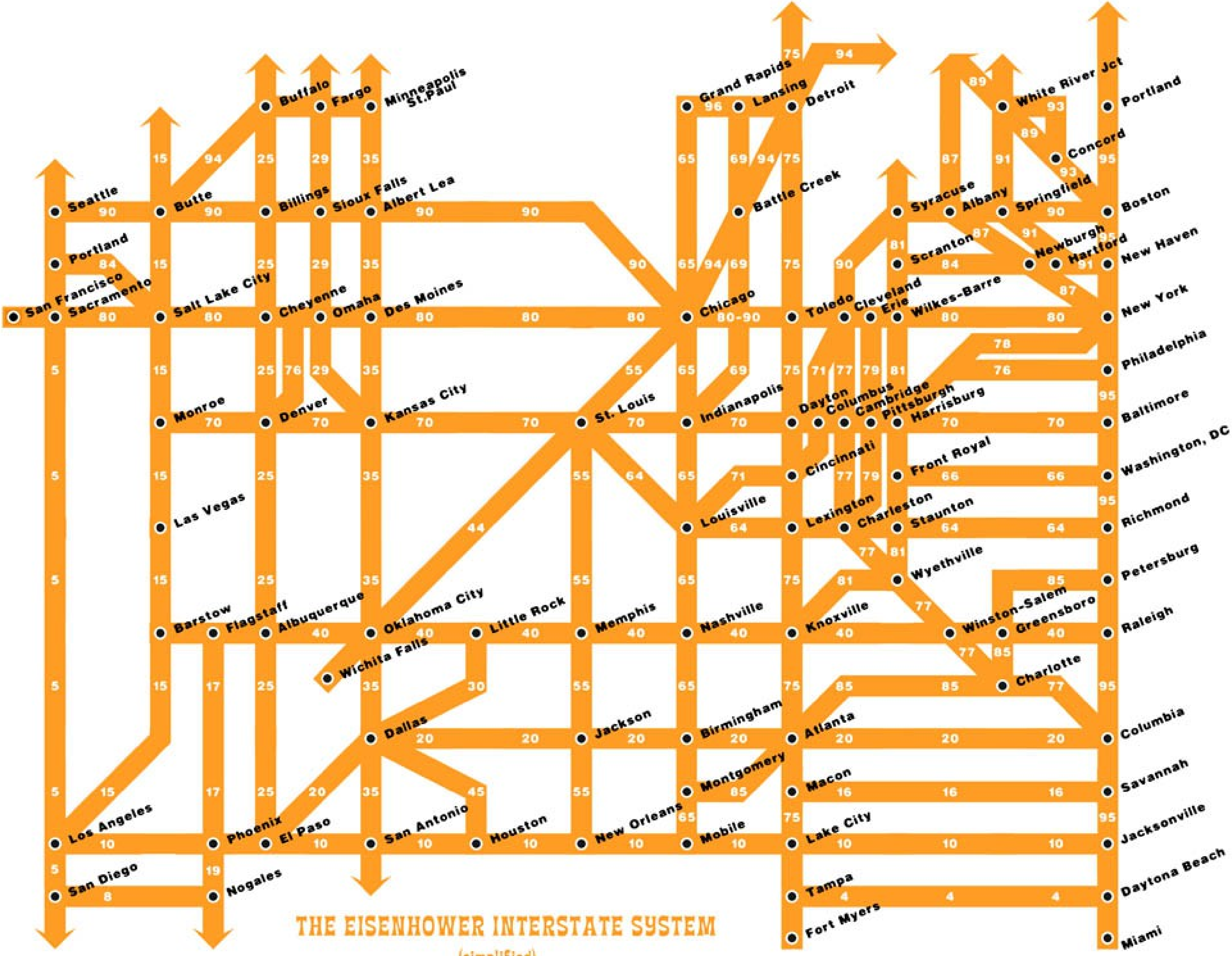
Using our Formalisms

- Let $G = (V, E)$ be an (undirected) graph.
- Intuitively, two nodes are adjacent if they're linked by an edge.
- Formally speaking, we say that two nodes $u, v \in V$ are ***adjacent*** if we have $\{u, v\} \in E$.
- There isn't an analogous notion for directed graphs. We usually just say "there's an edge from u to v" as a way of reading $(u, v) \in E$ aloud.

New Stuff!

Walks, Paths, and Reachability



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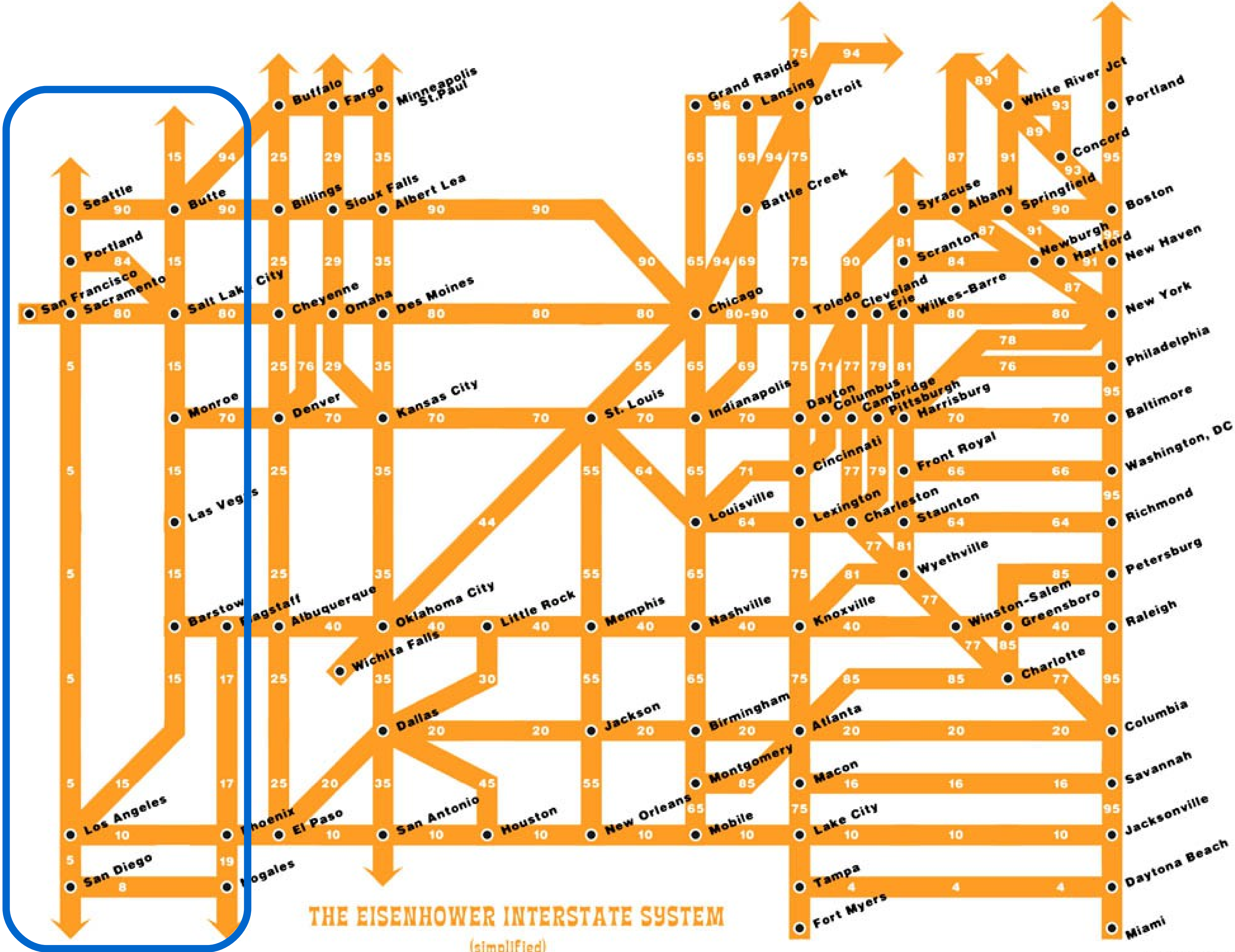

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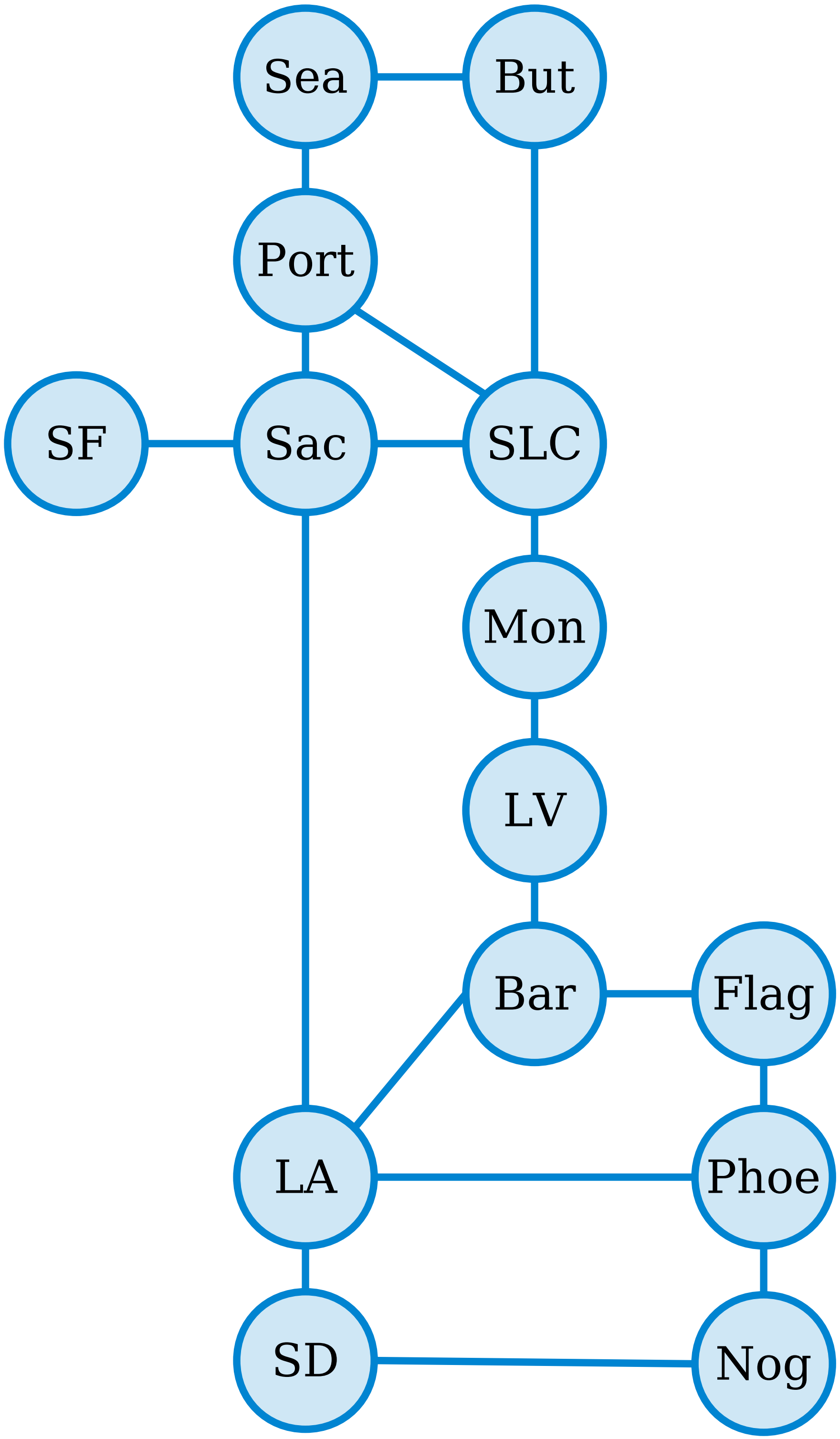
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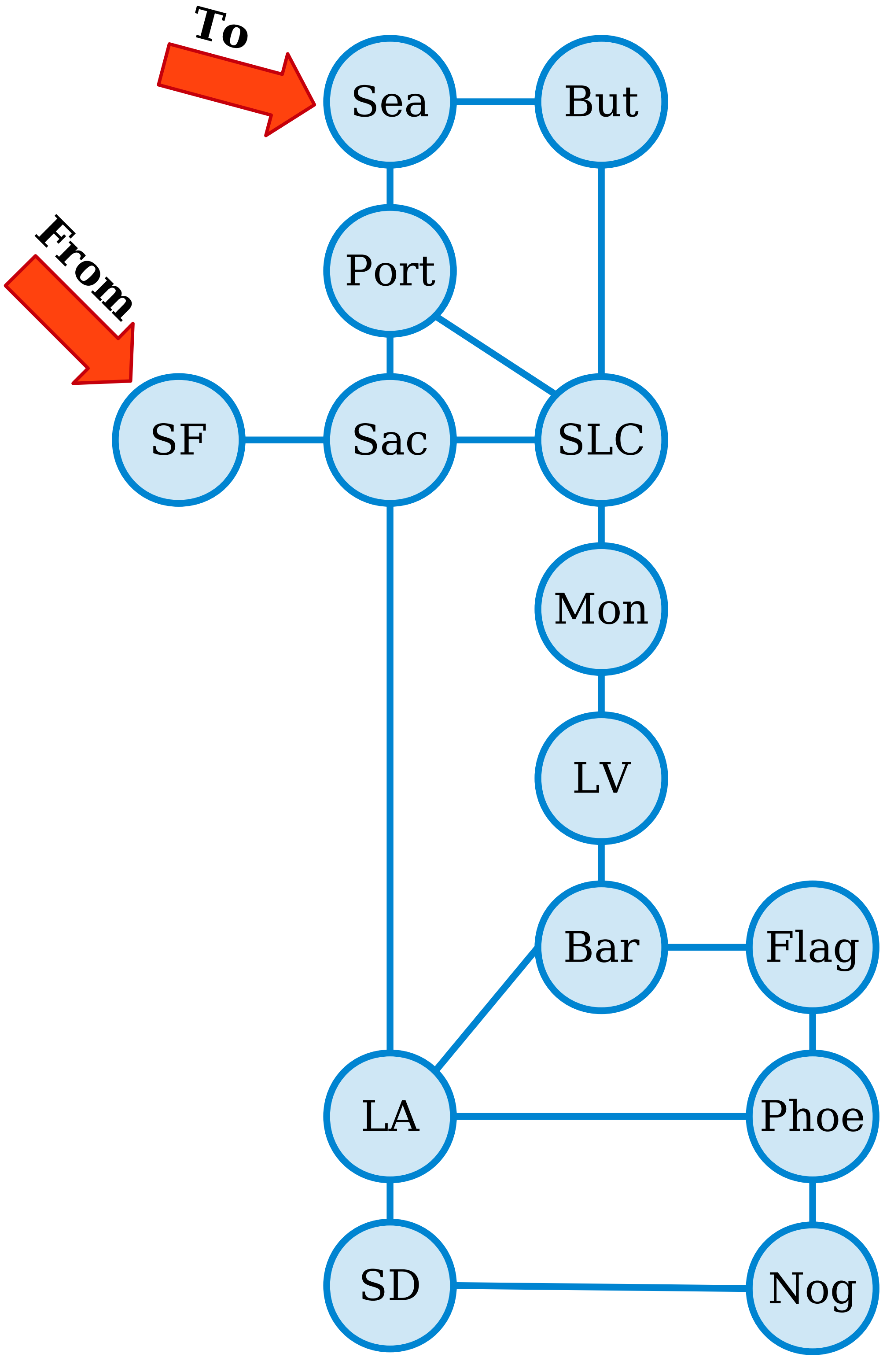
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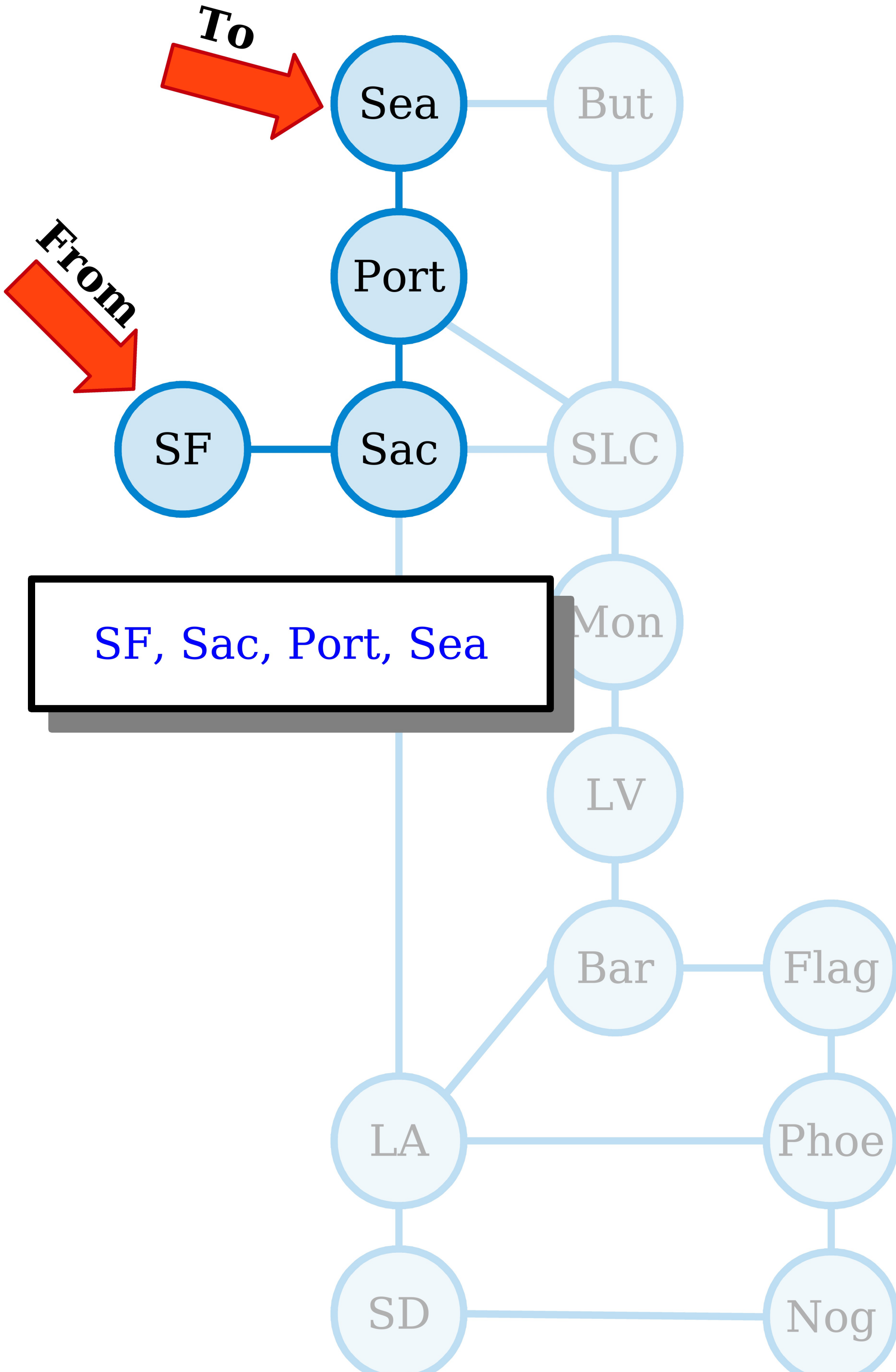
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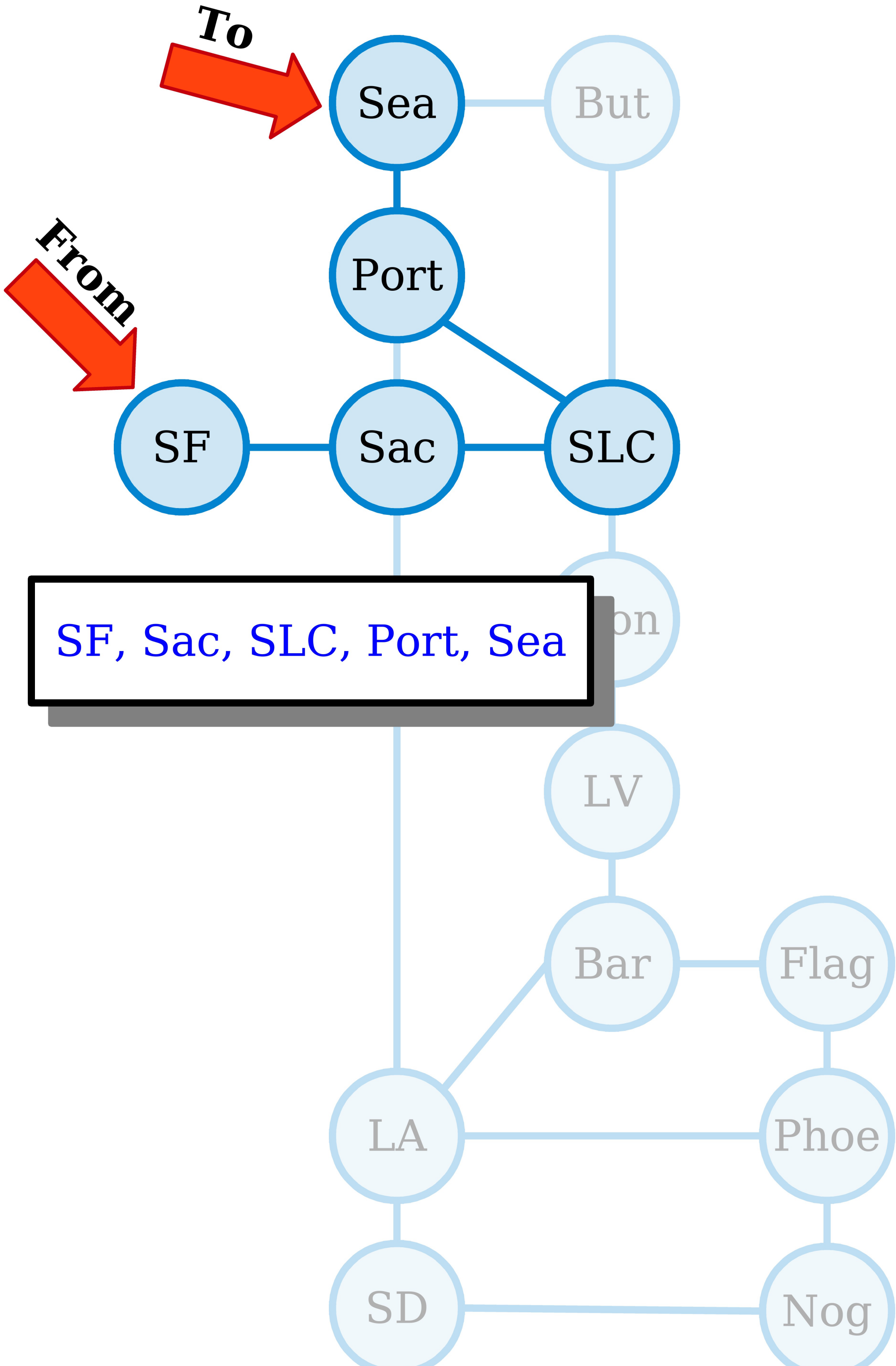

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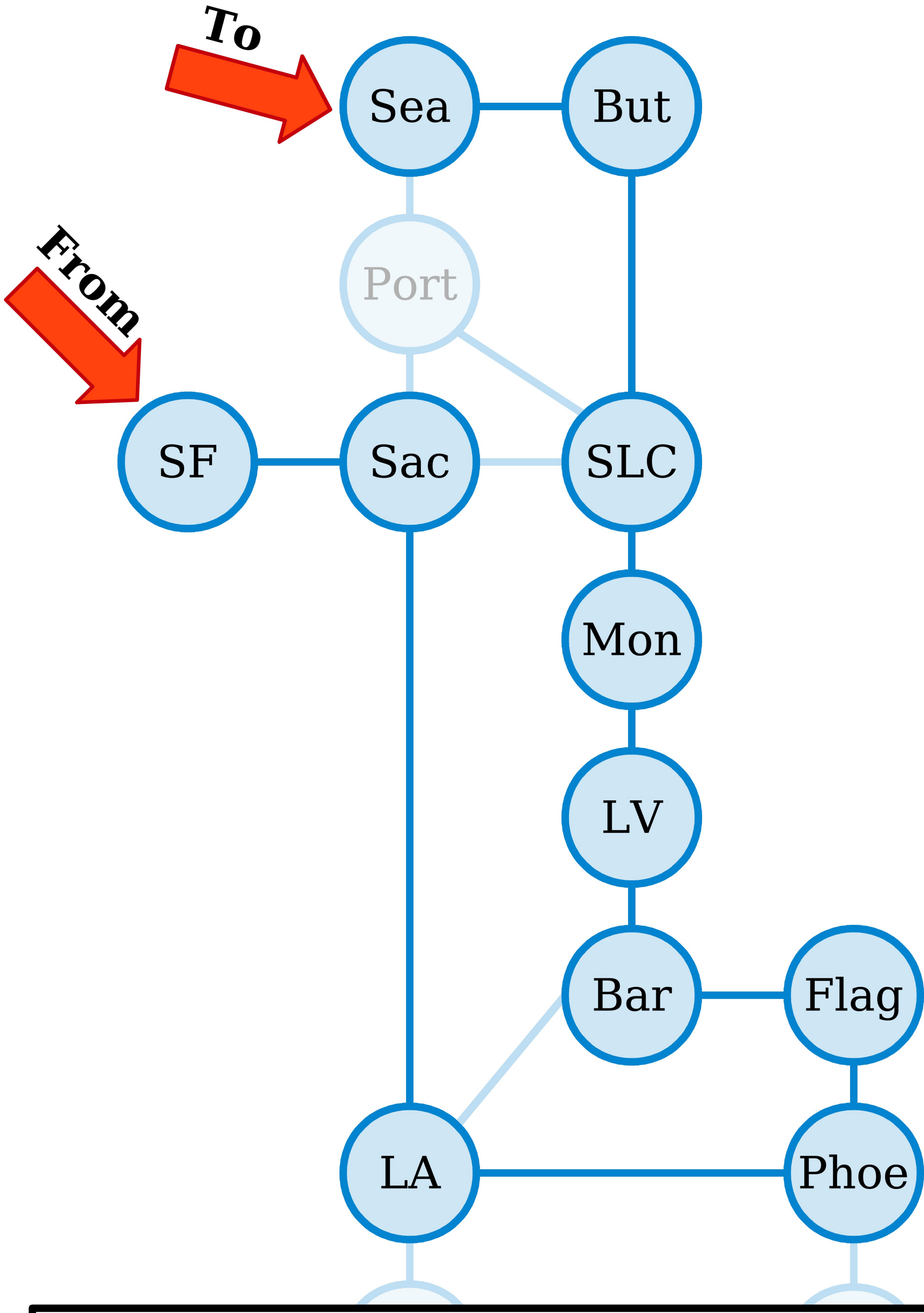
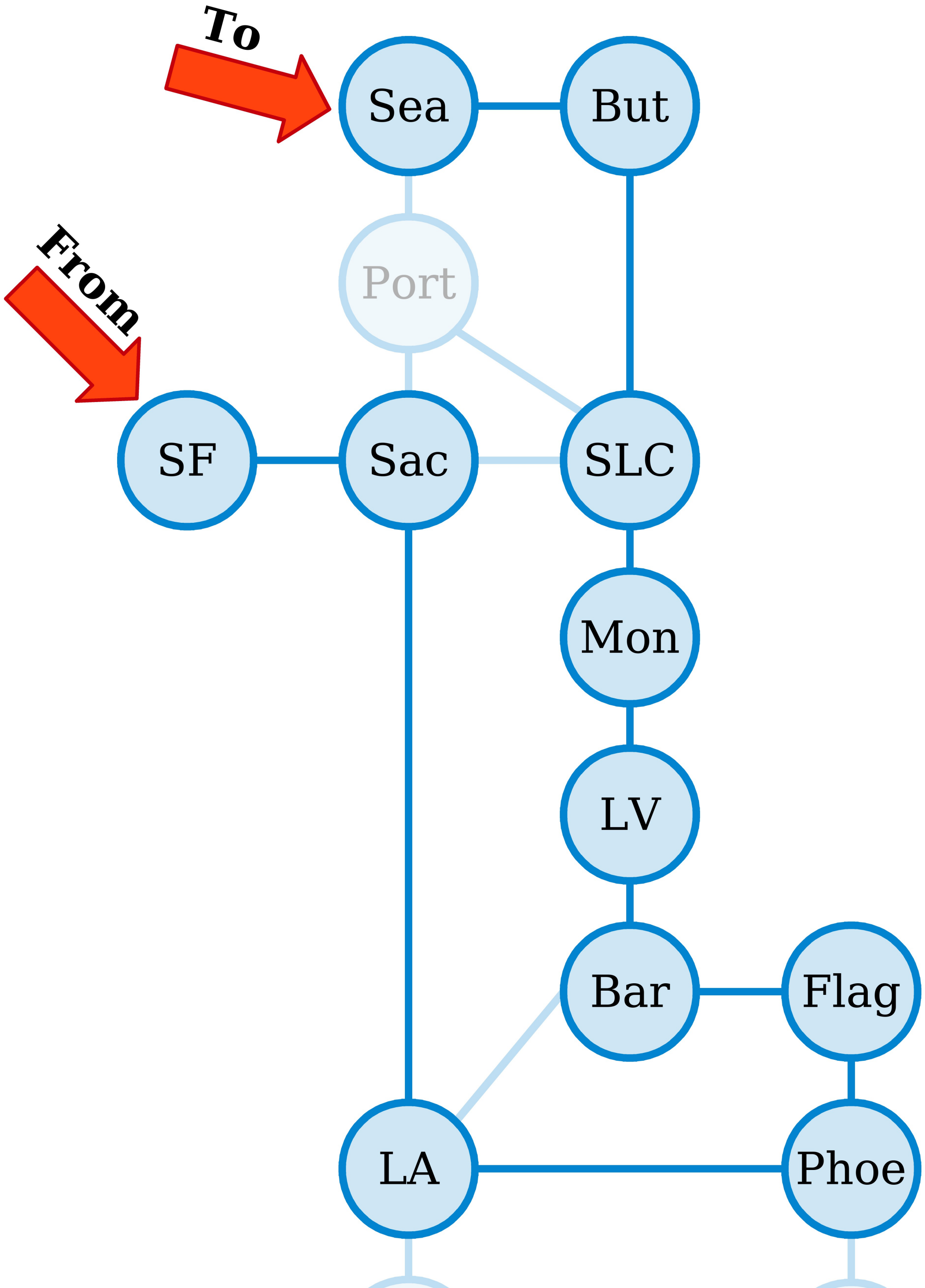


SF, Sac, LA, Phoe, Flag, Bar, LV, Mon, SLC, But, Sea



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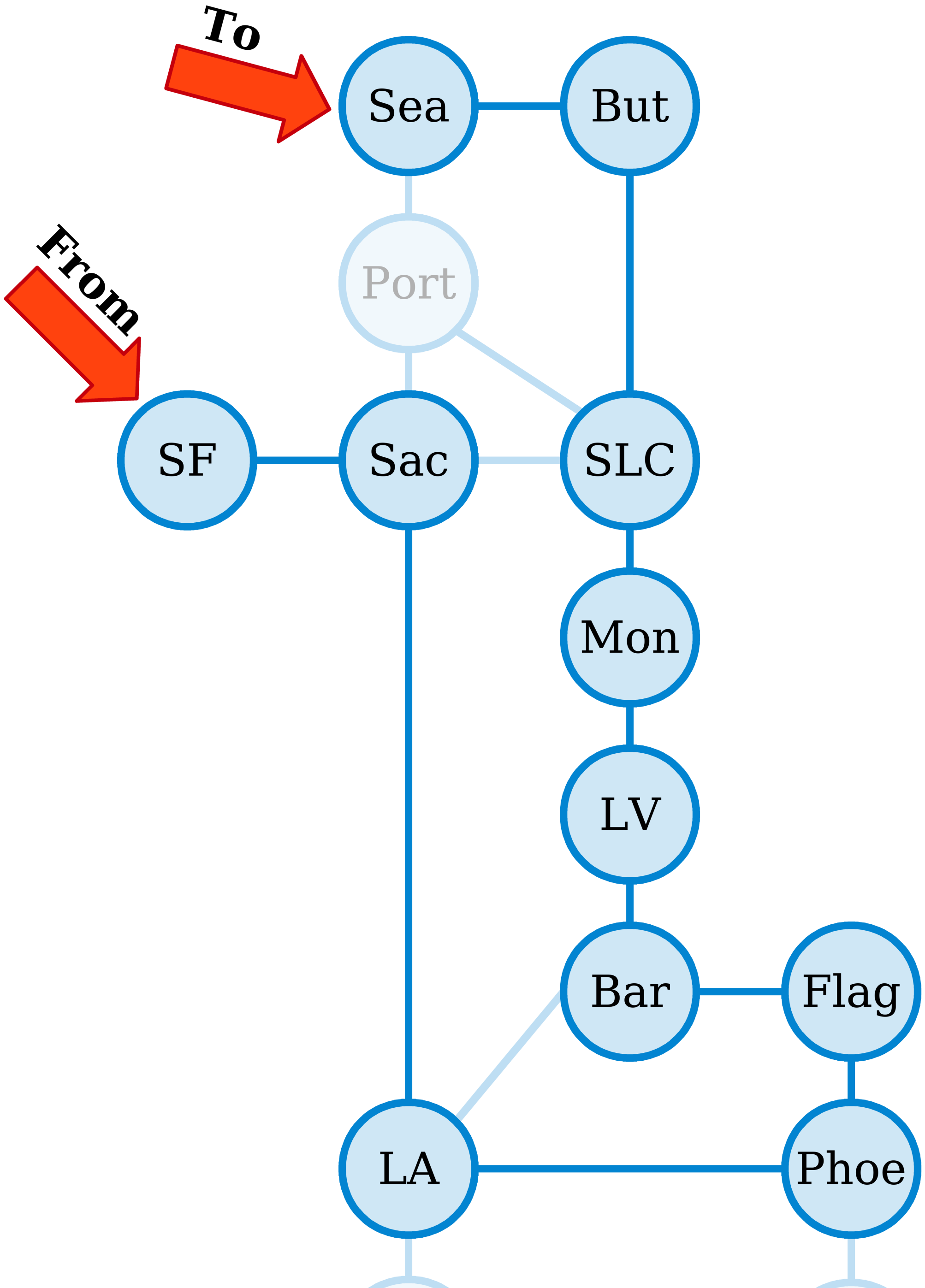

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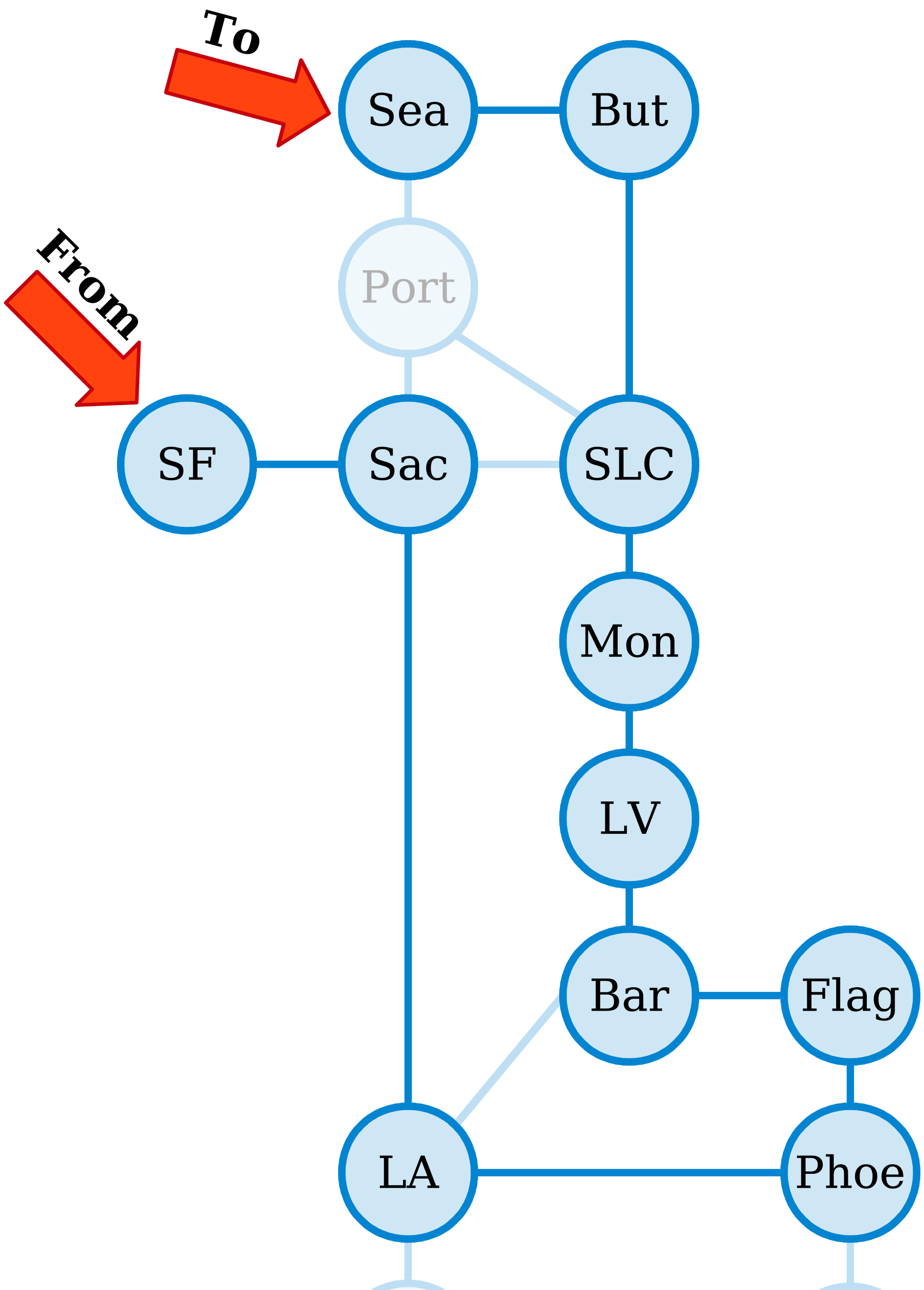

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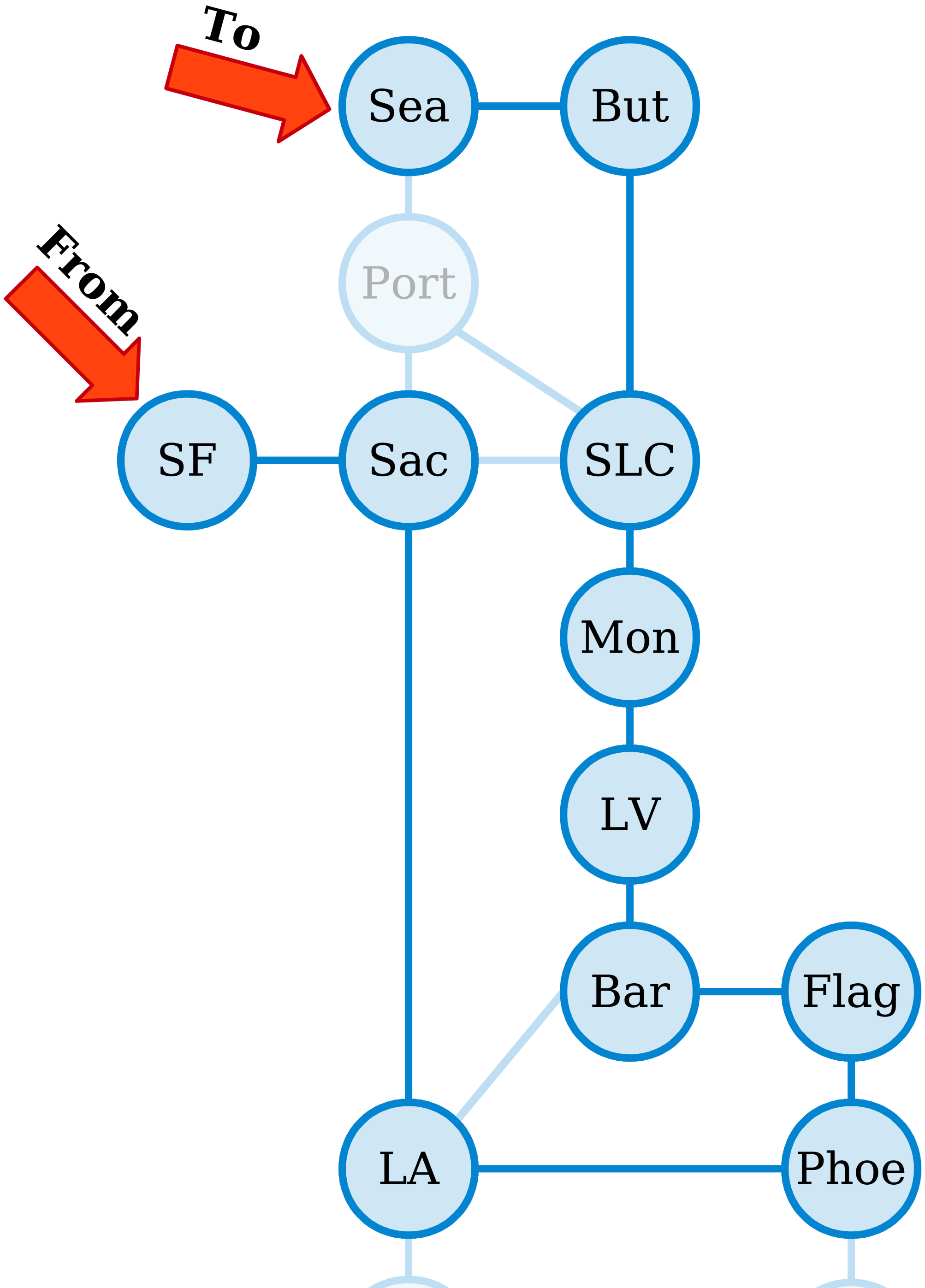

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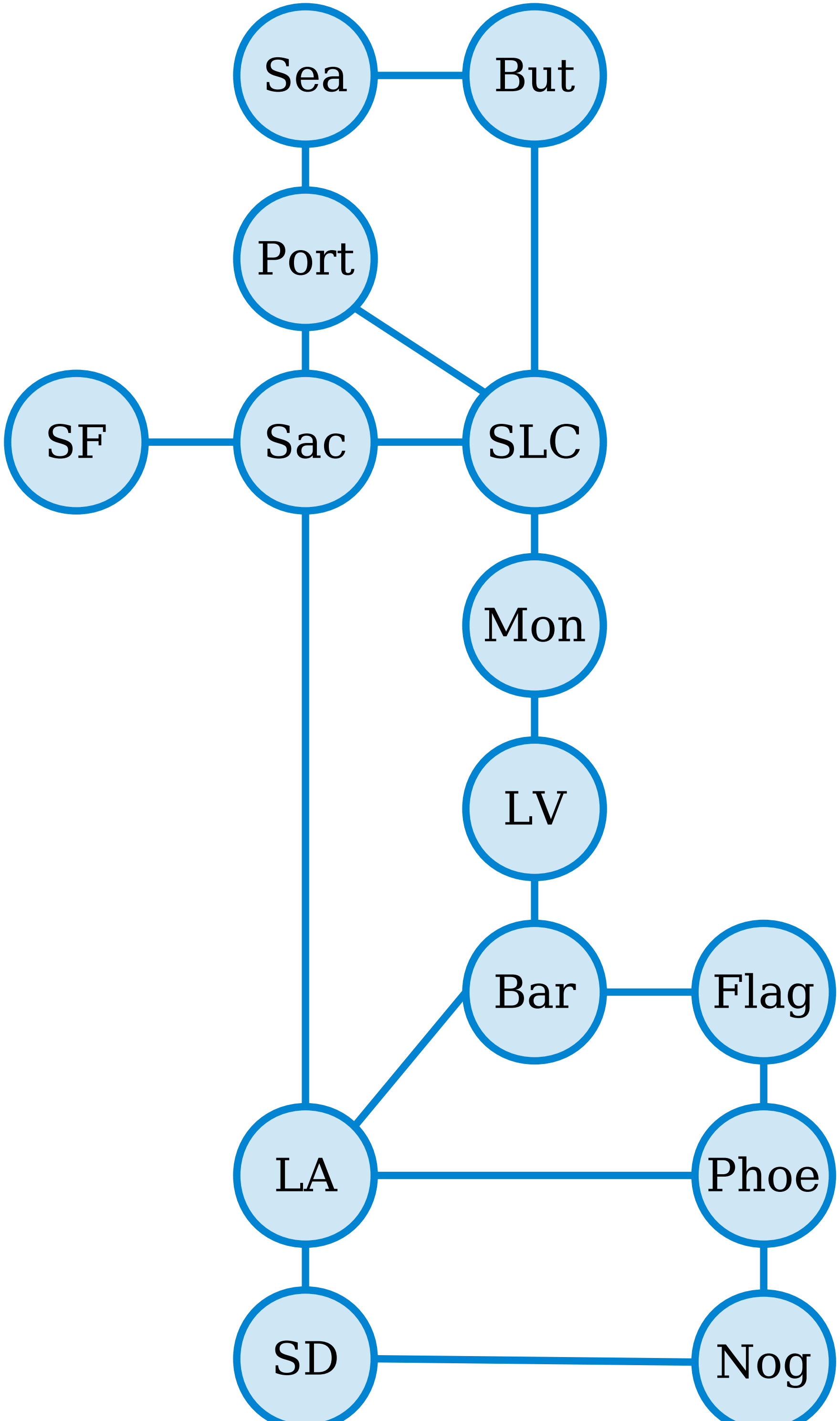

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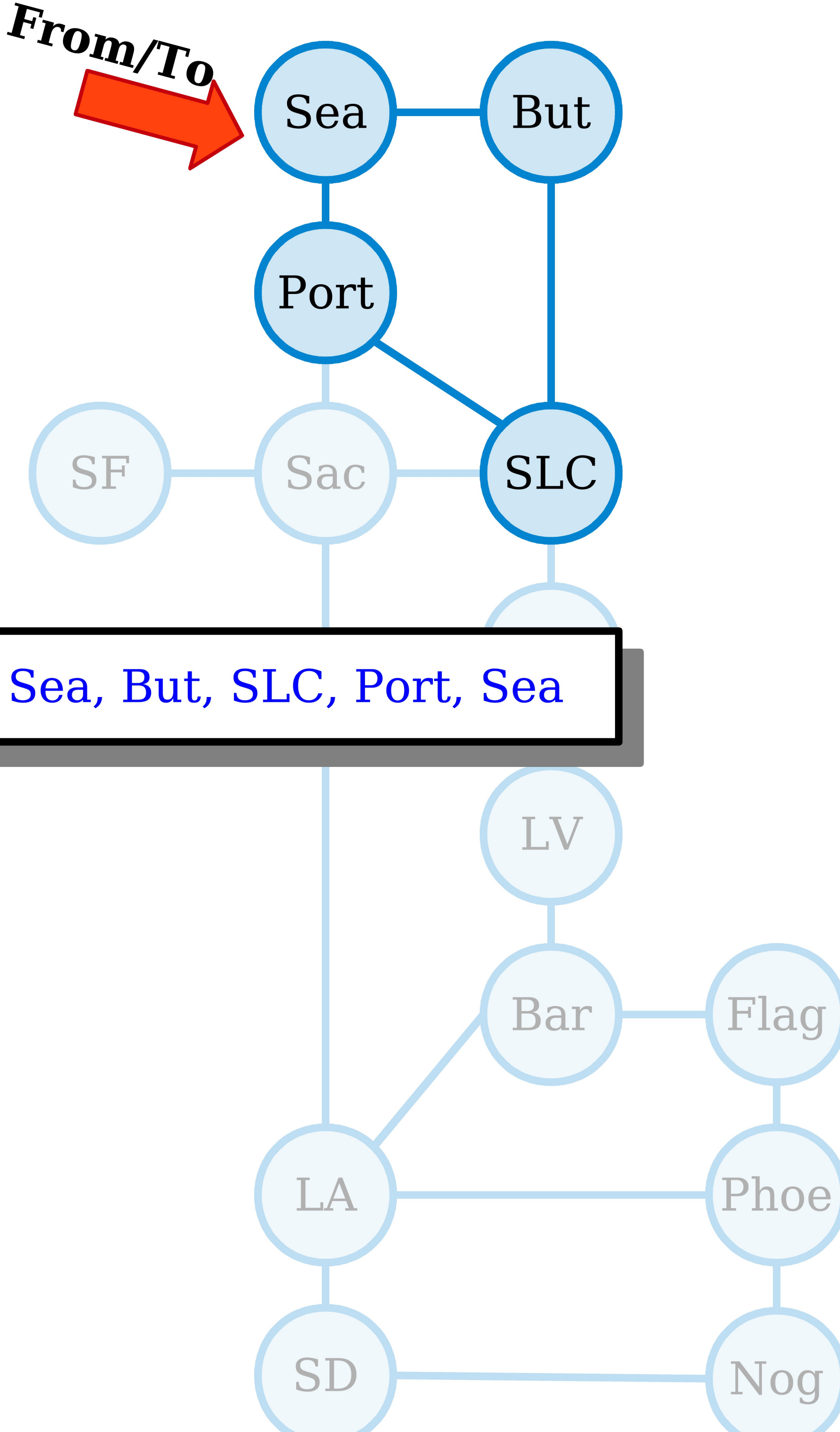
Go to PollEv.com/cs103spr25



Sea, But, SLC, Port, Sea

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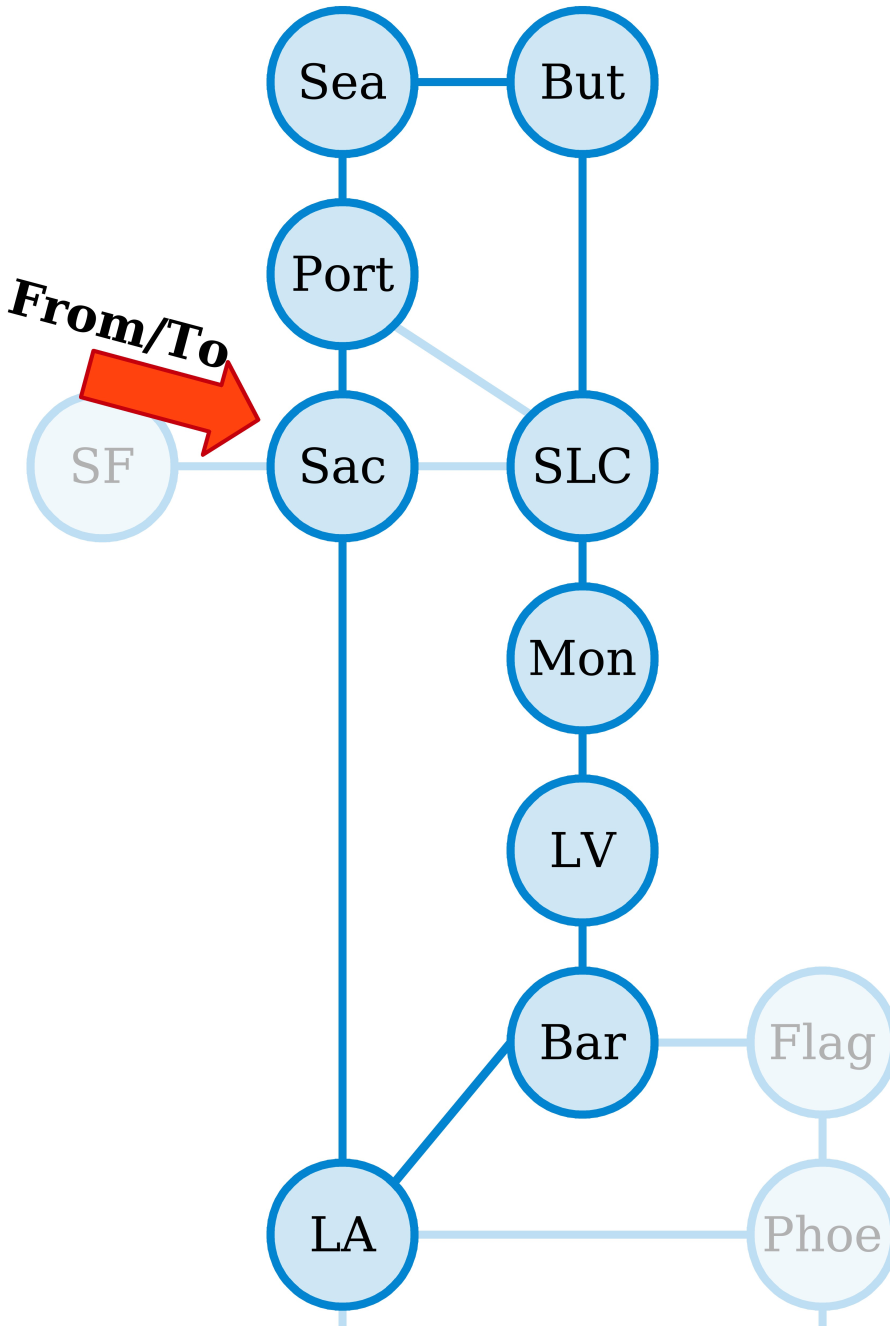

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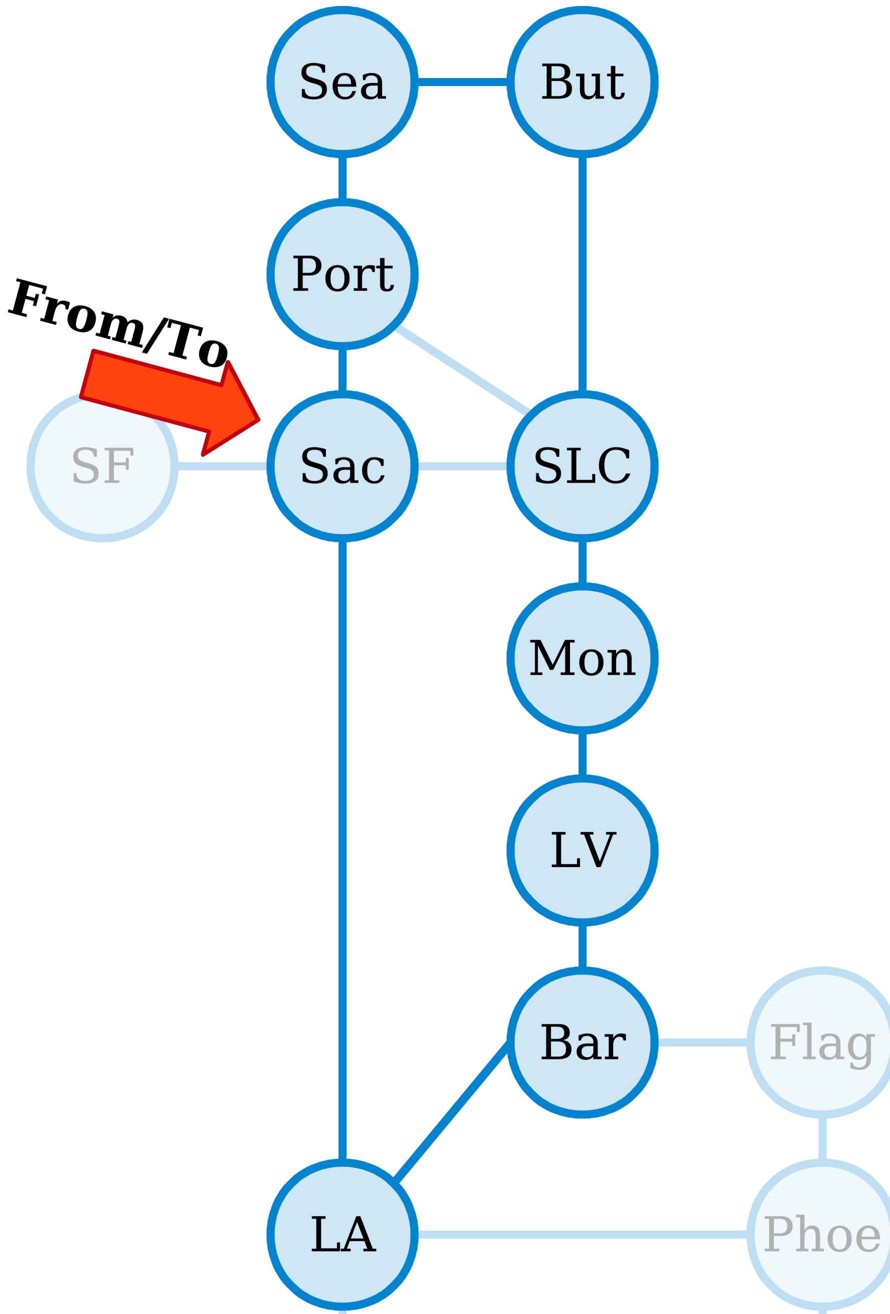

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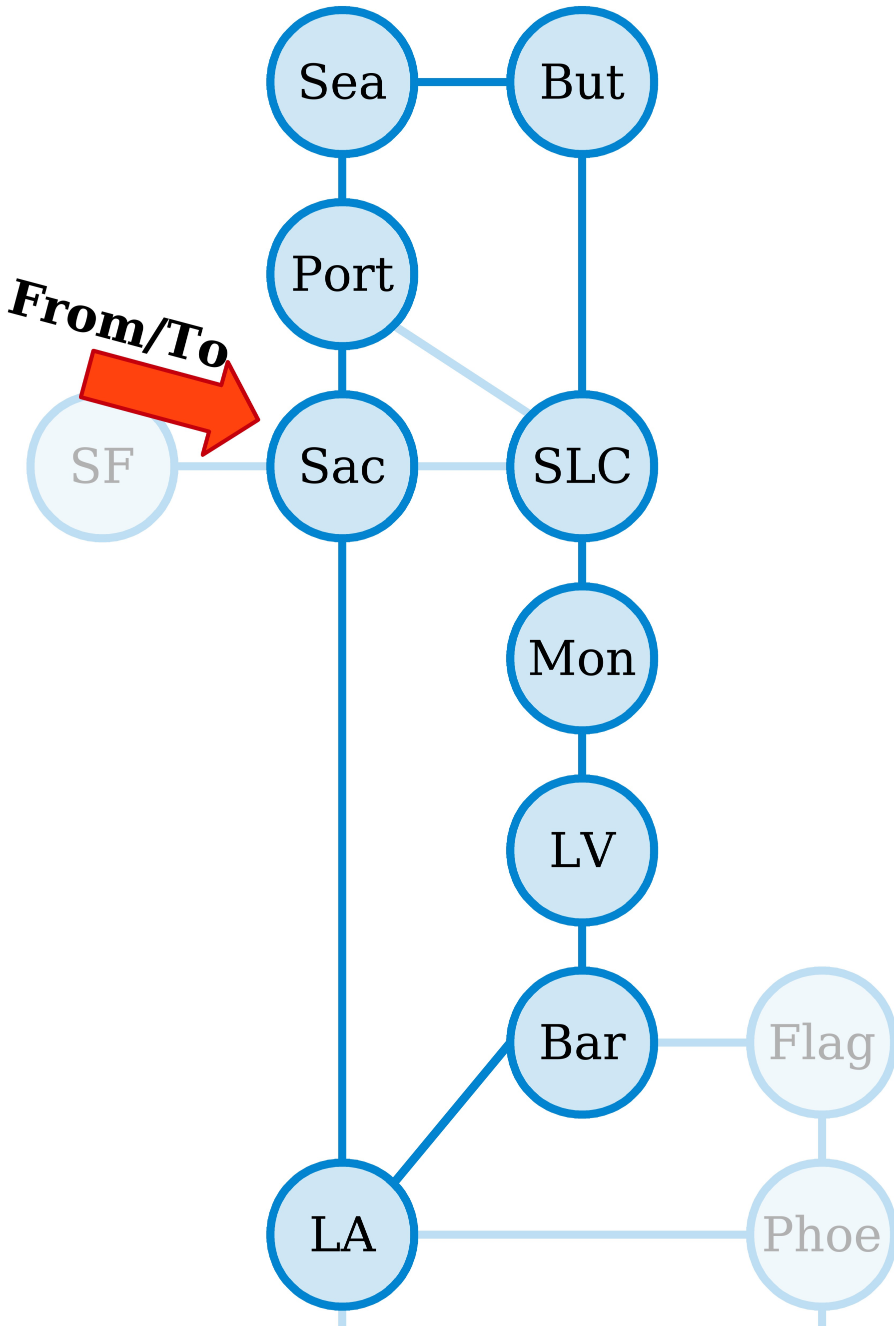

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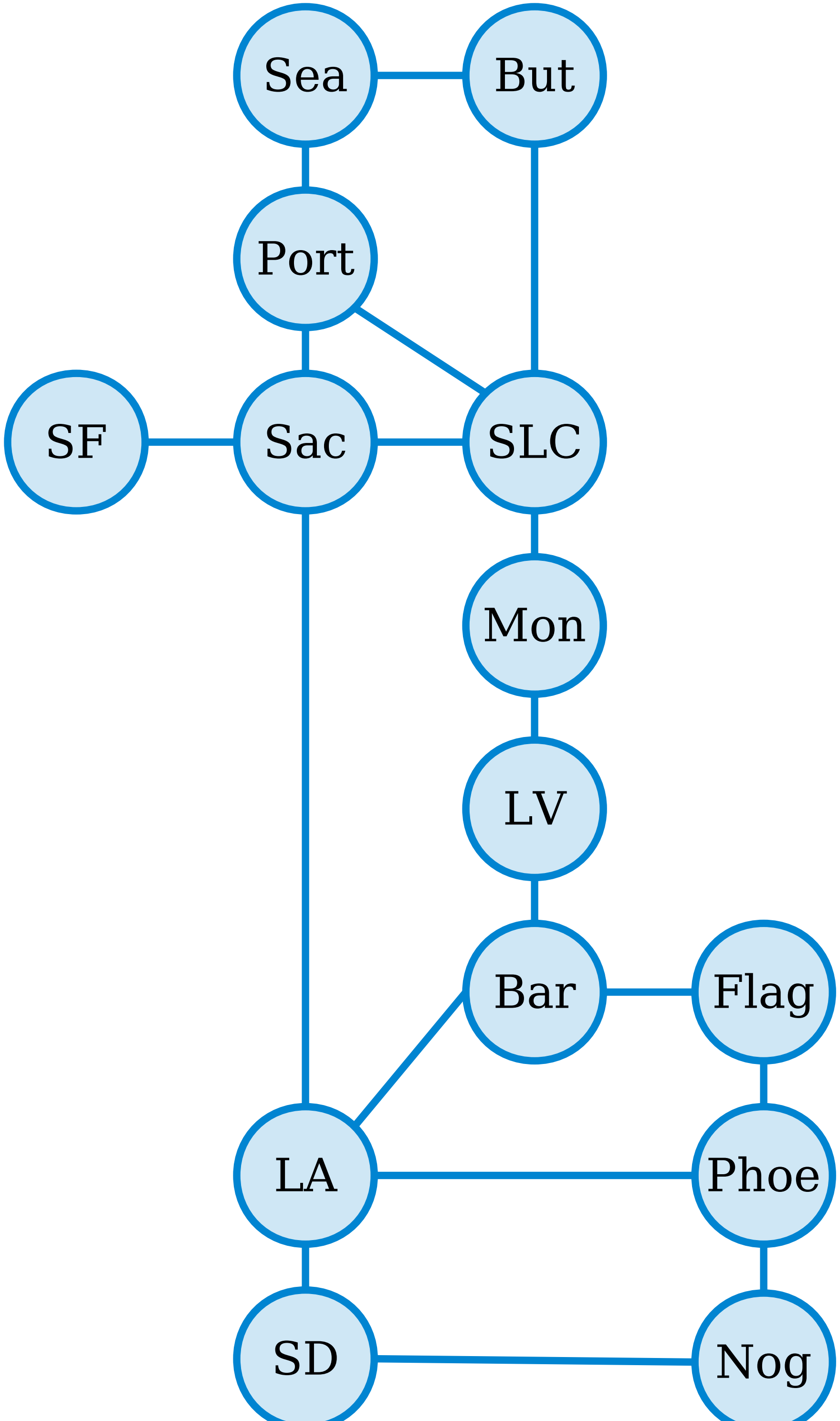
Sac, Port, Sea, But, SLC, Mon, LV, Bar, LA, Sac



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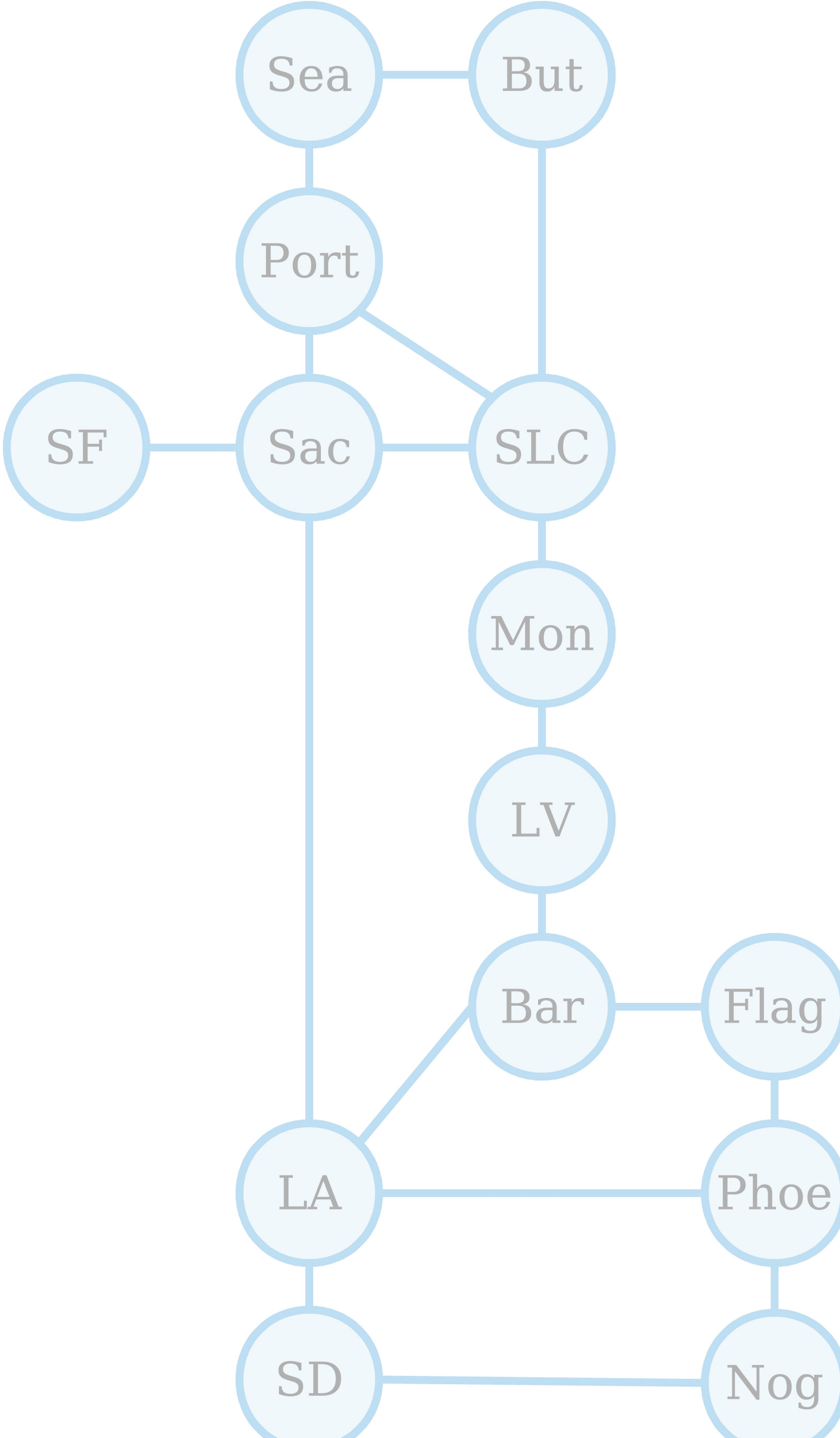

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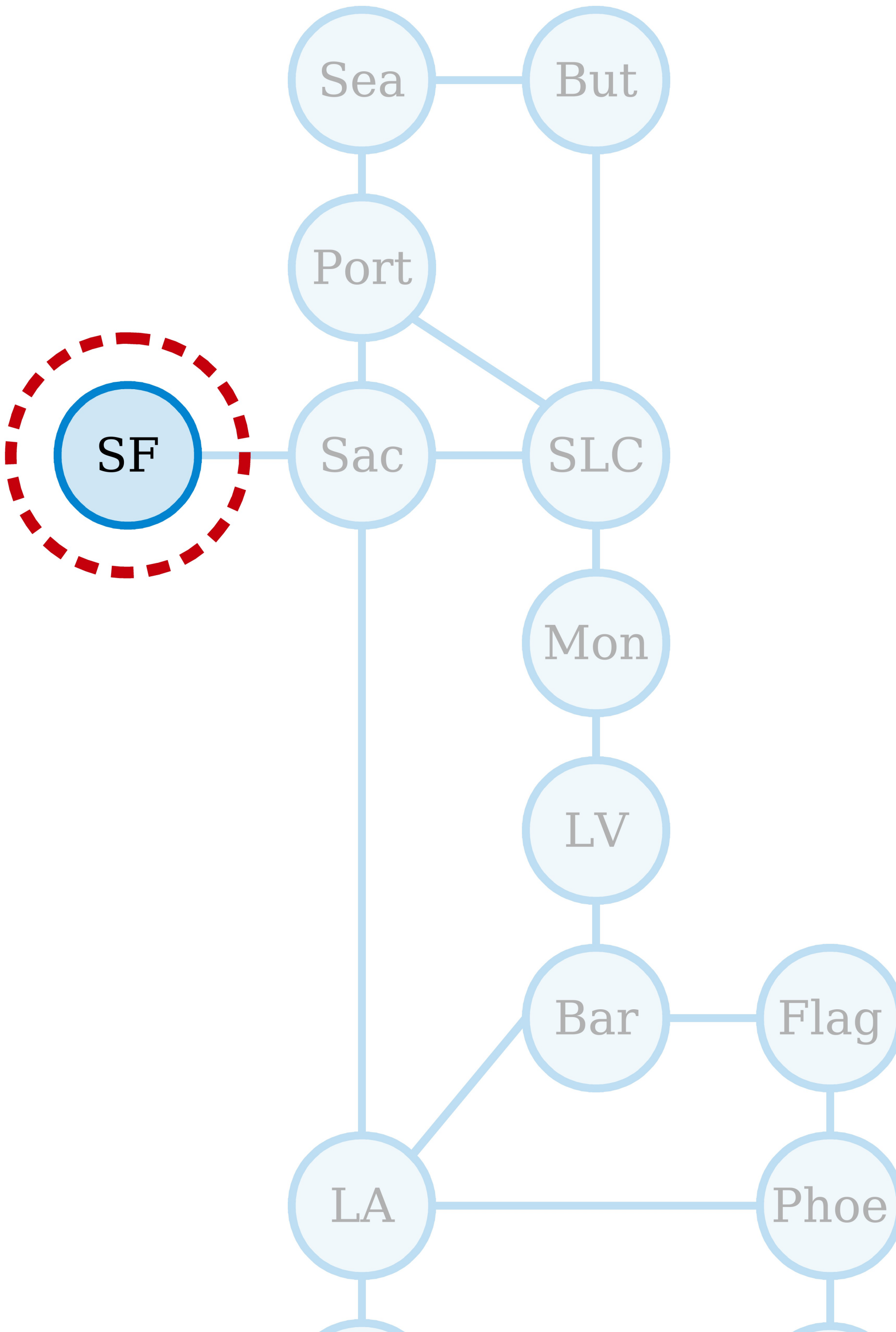

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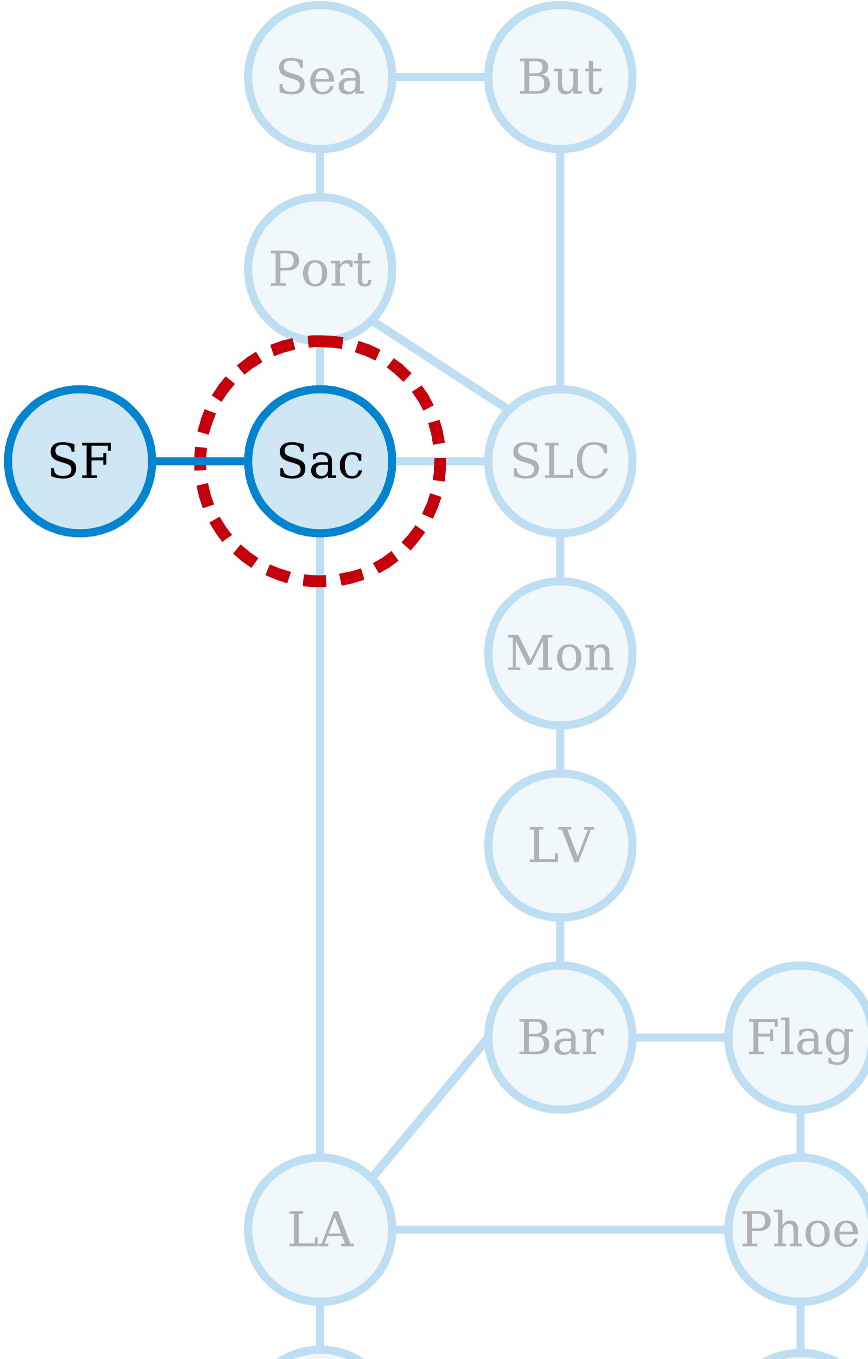

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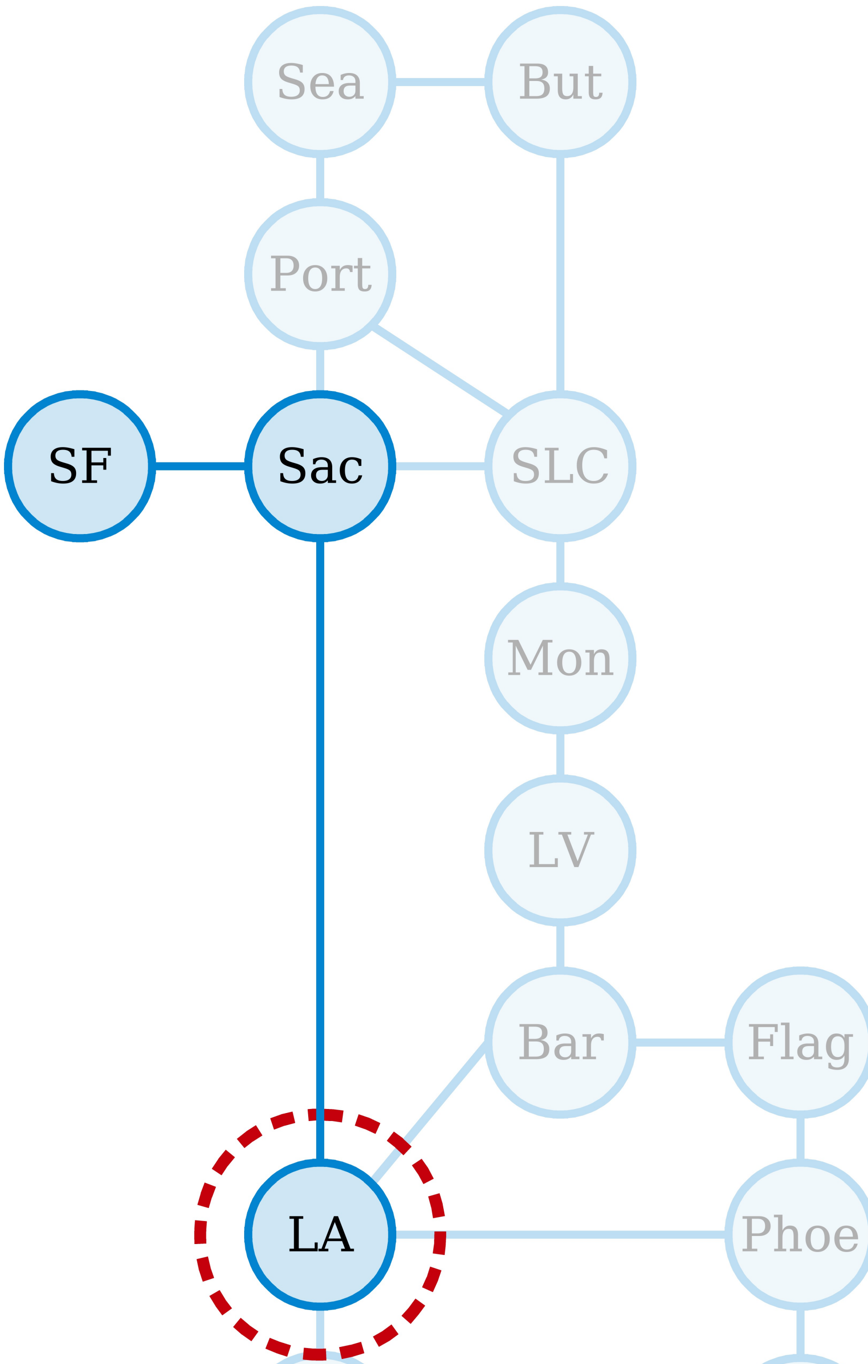

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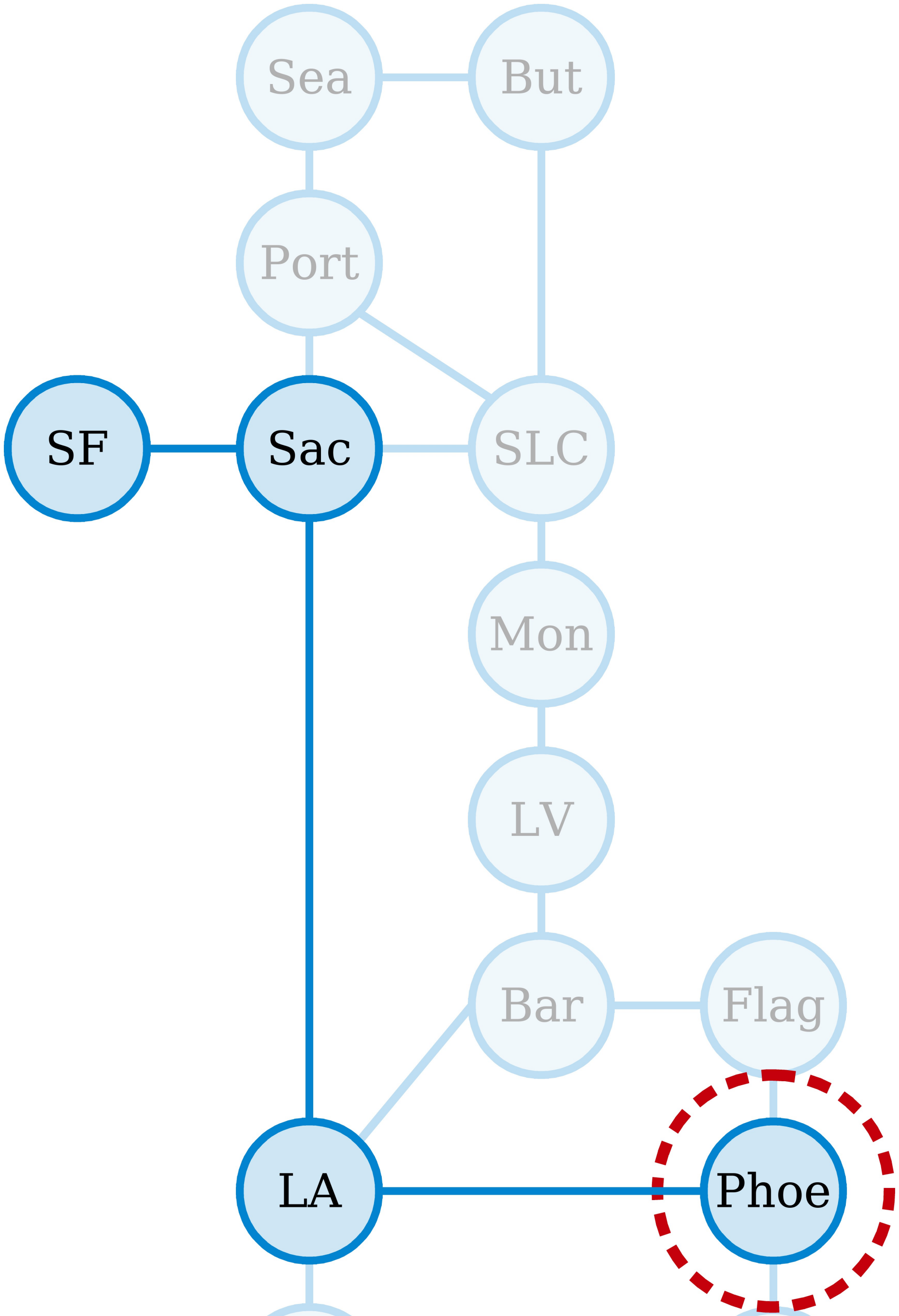

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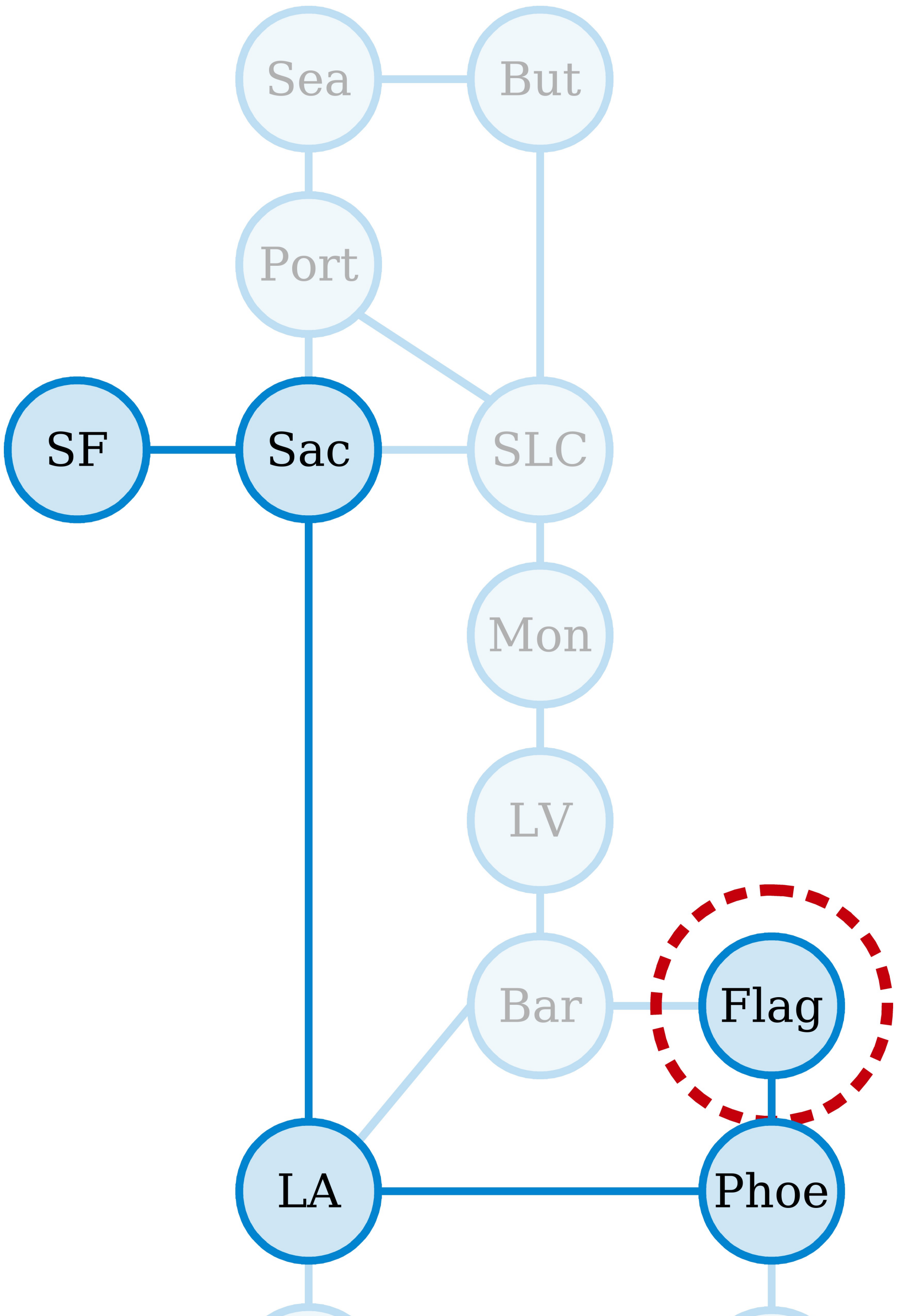
SF, Sac, LA, Phoe



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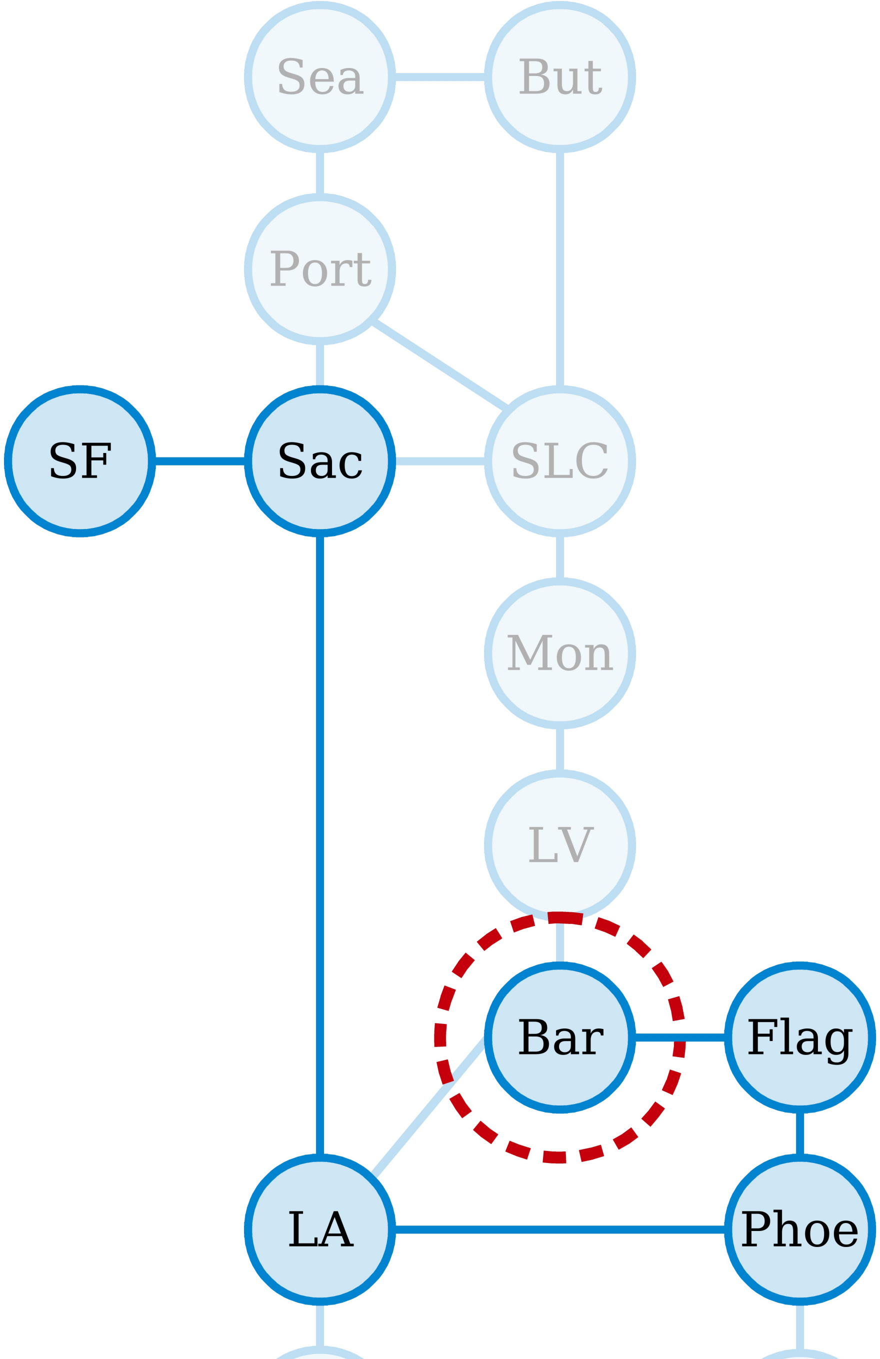
SF, Sac, LA, Phoe, Flag



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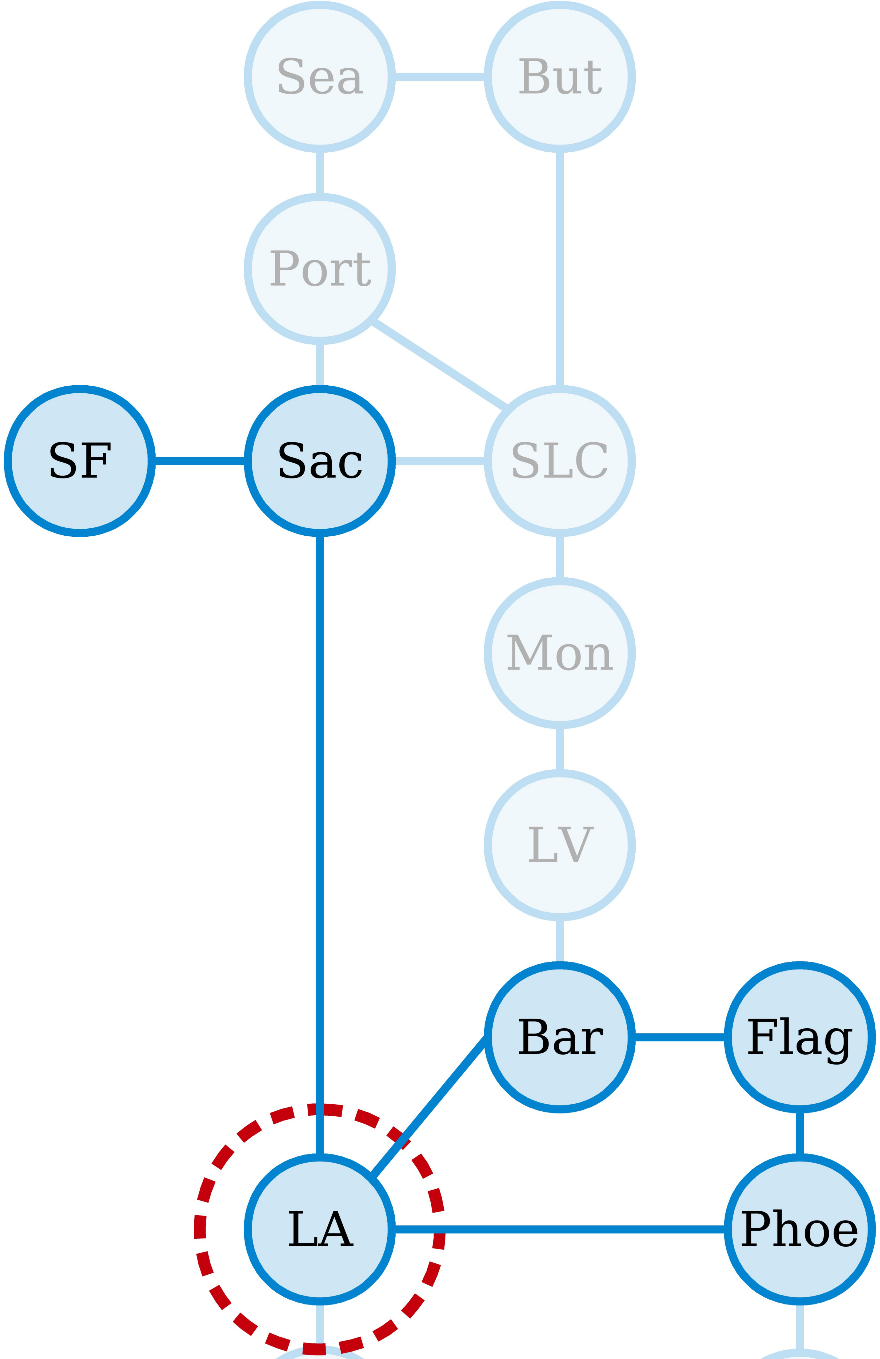

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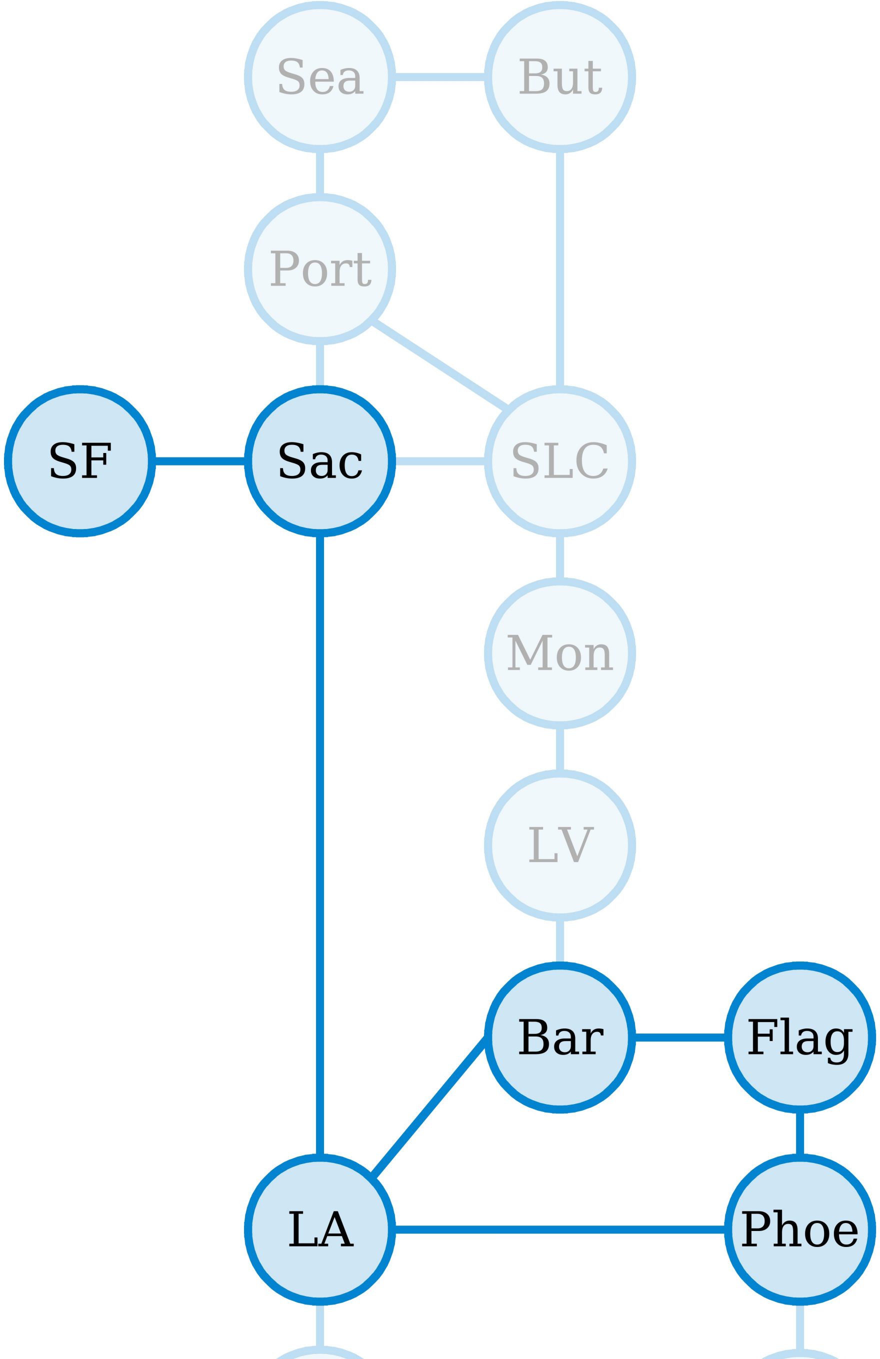

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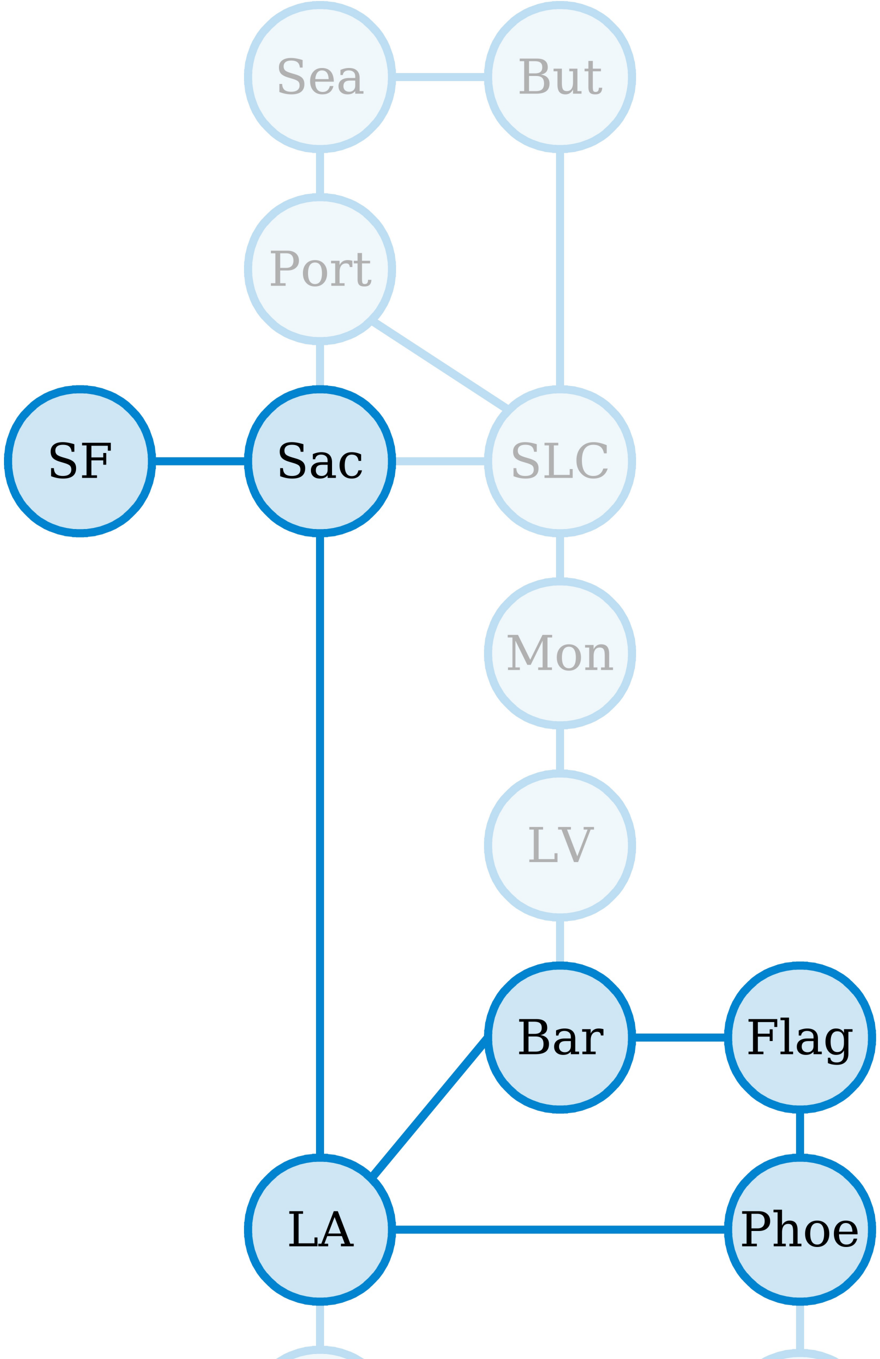

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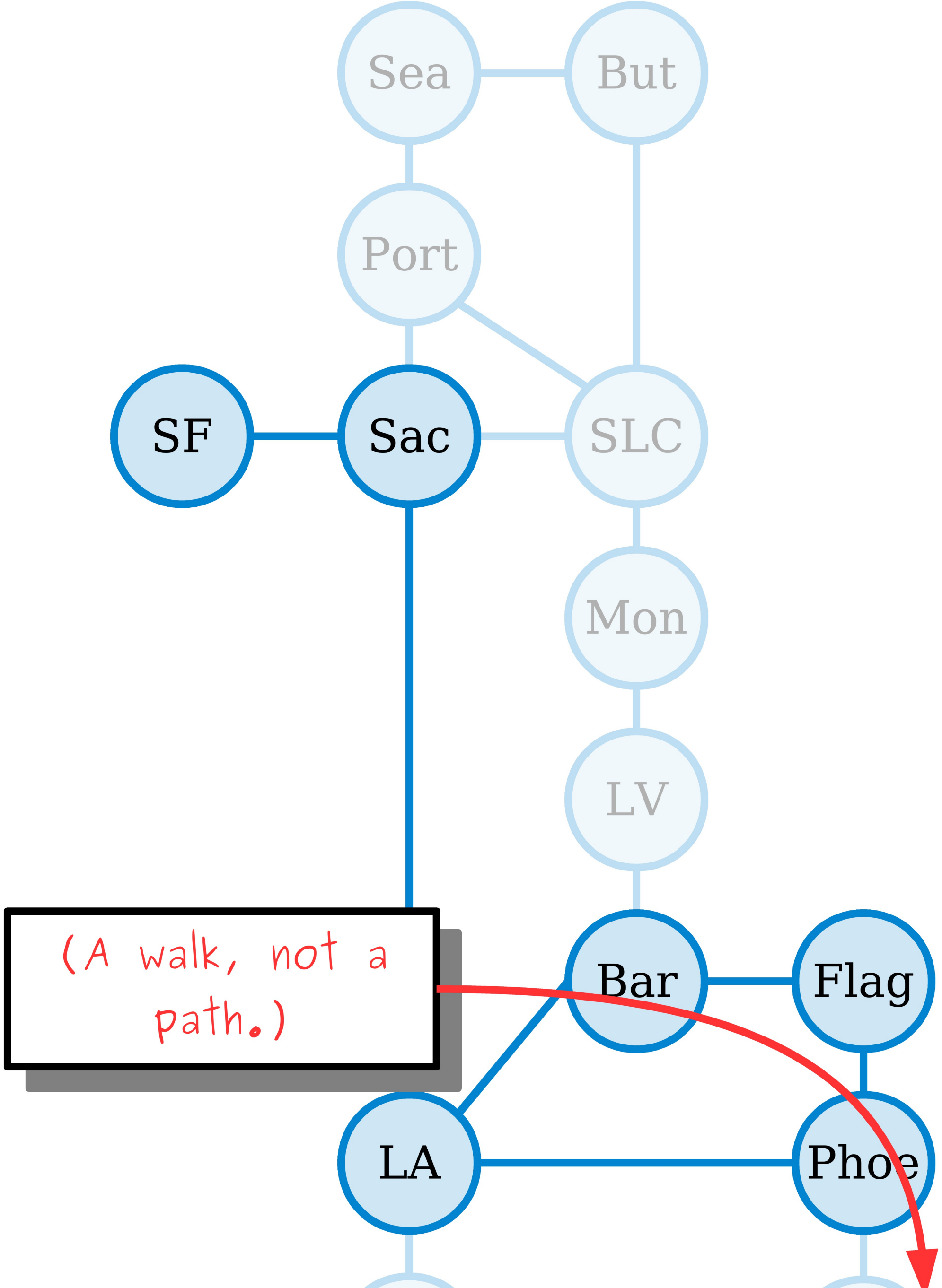

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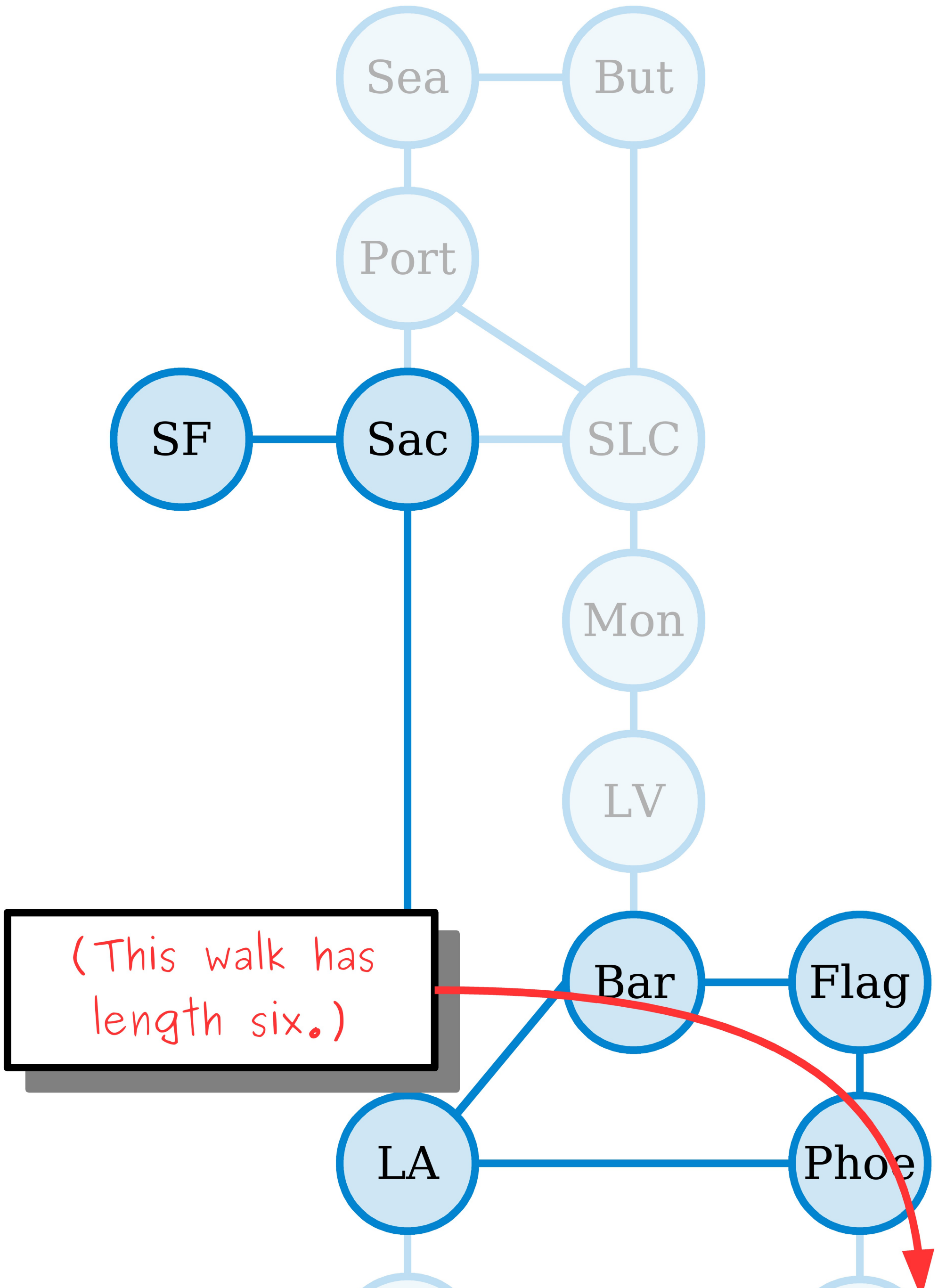

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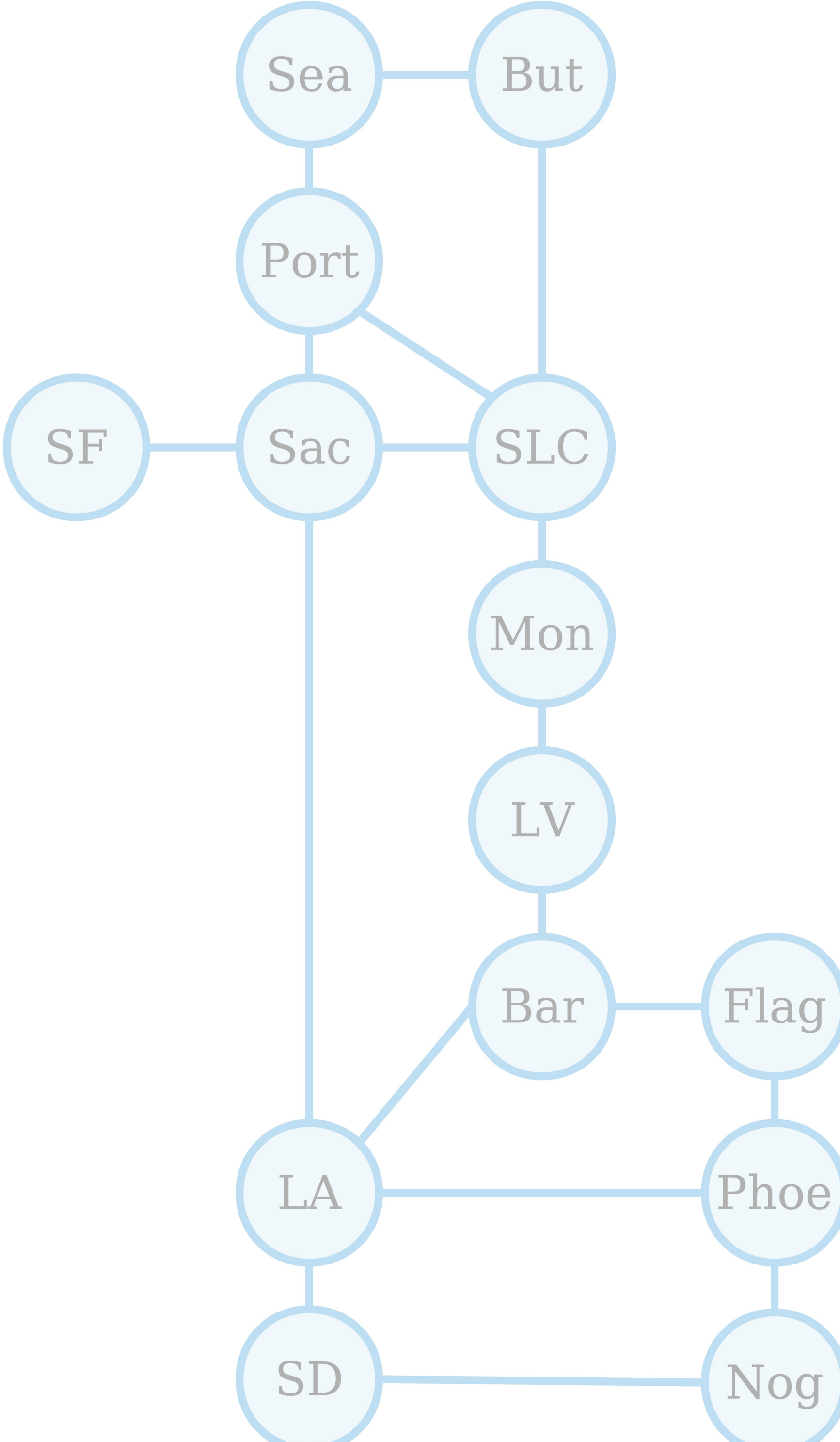

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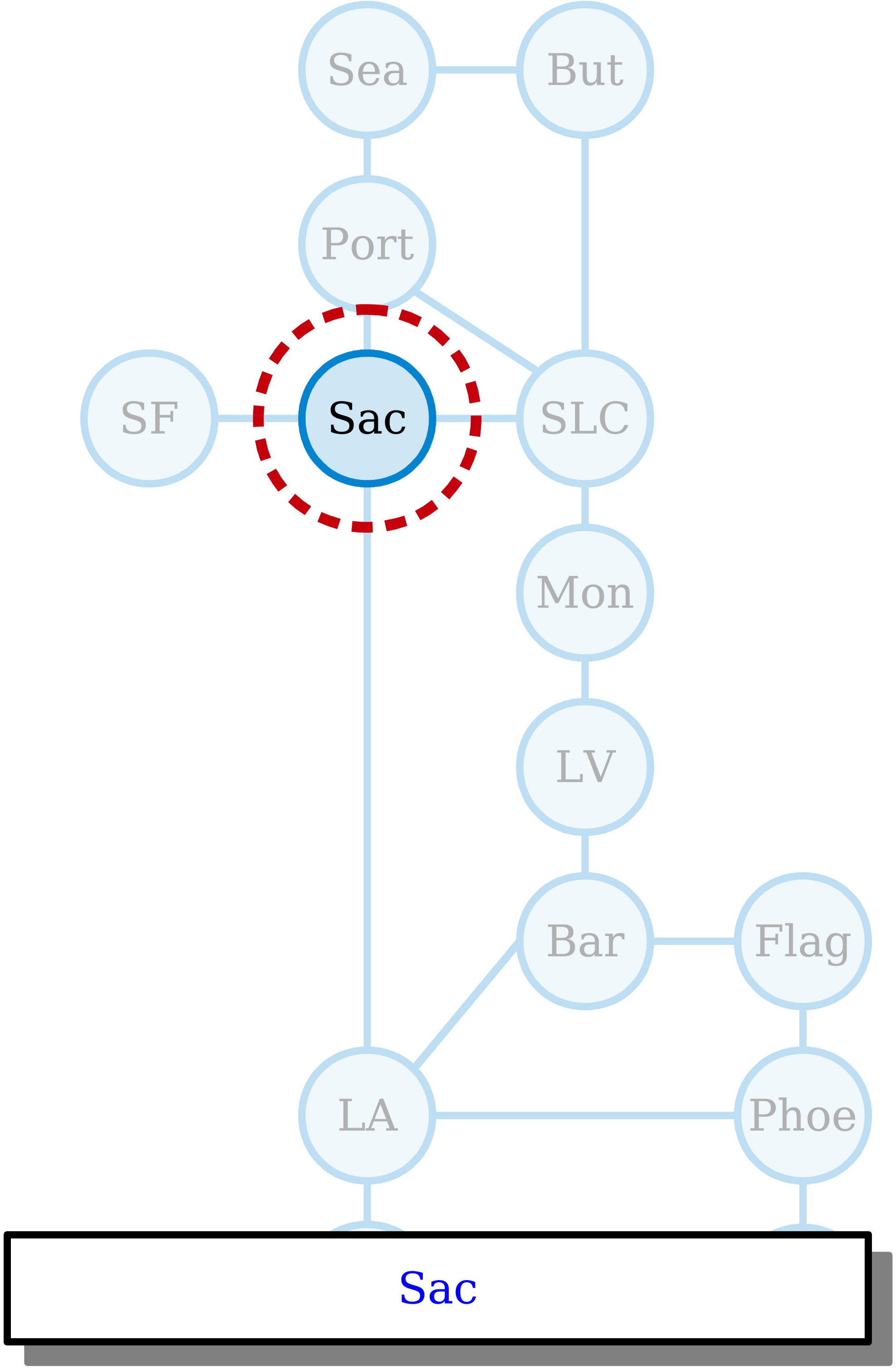

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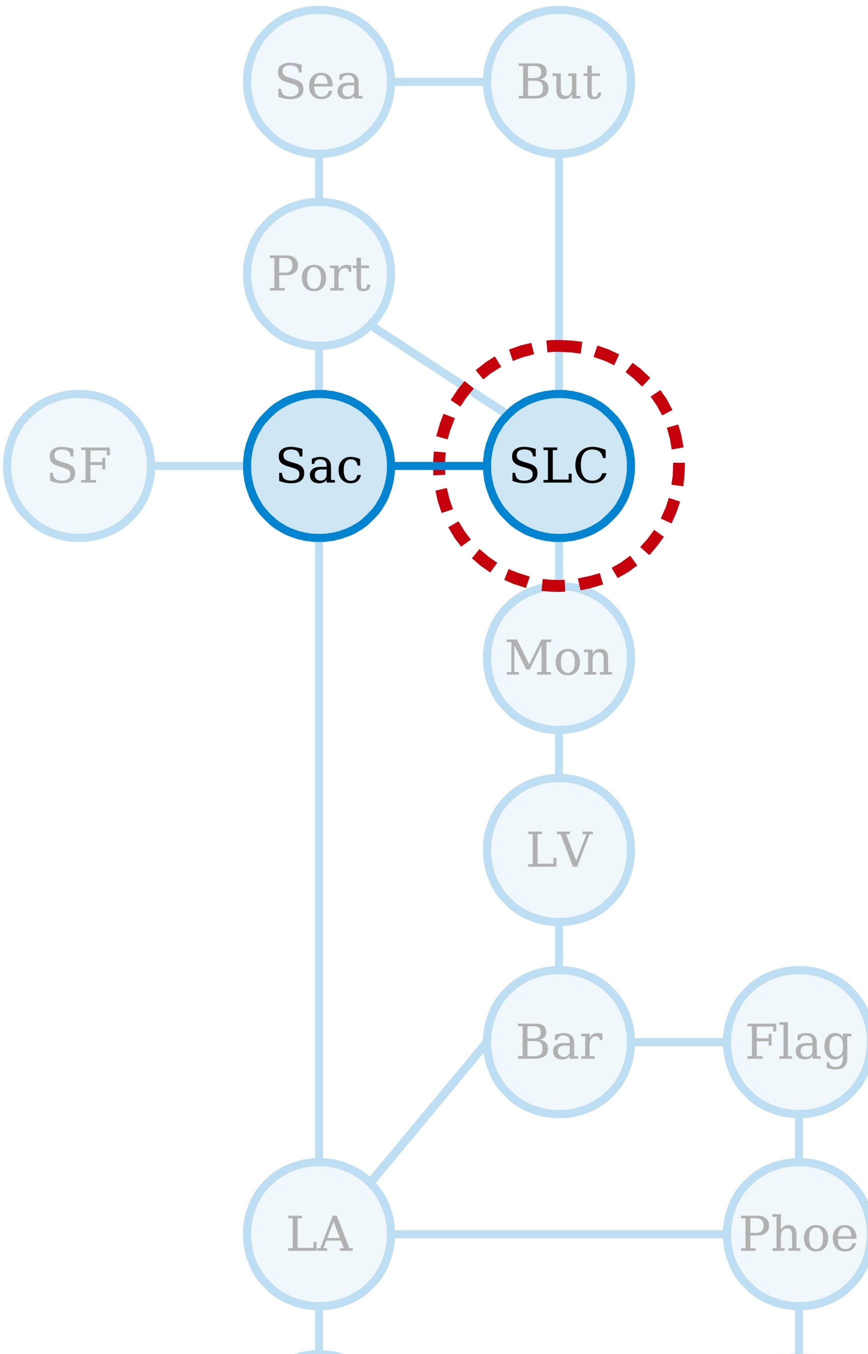

Sac, SLC

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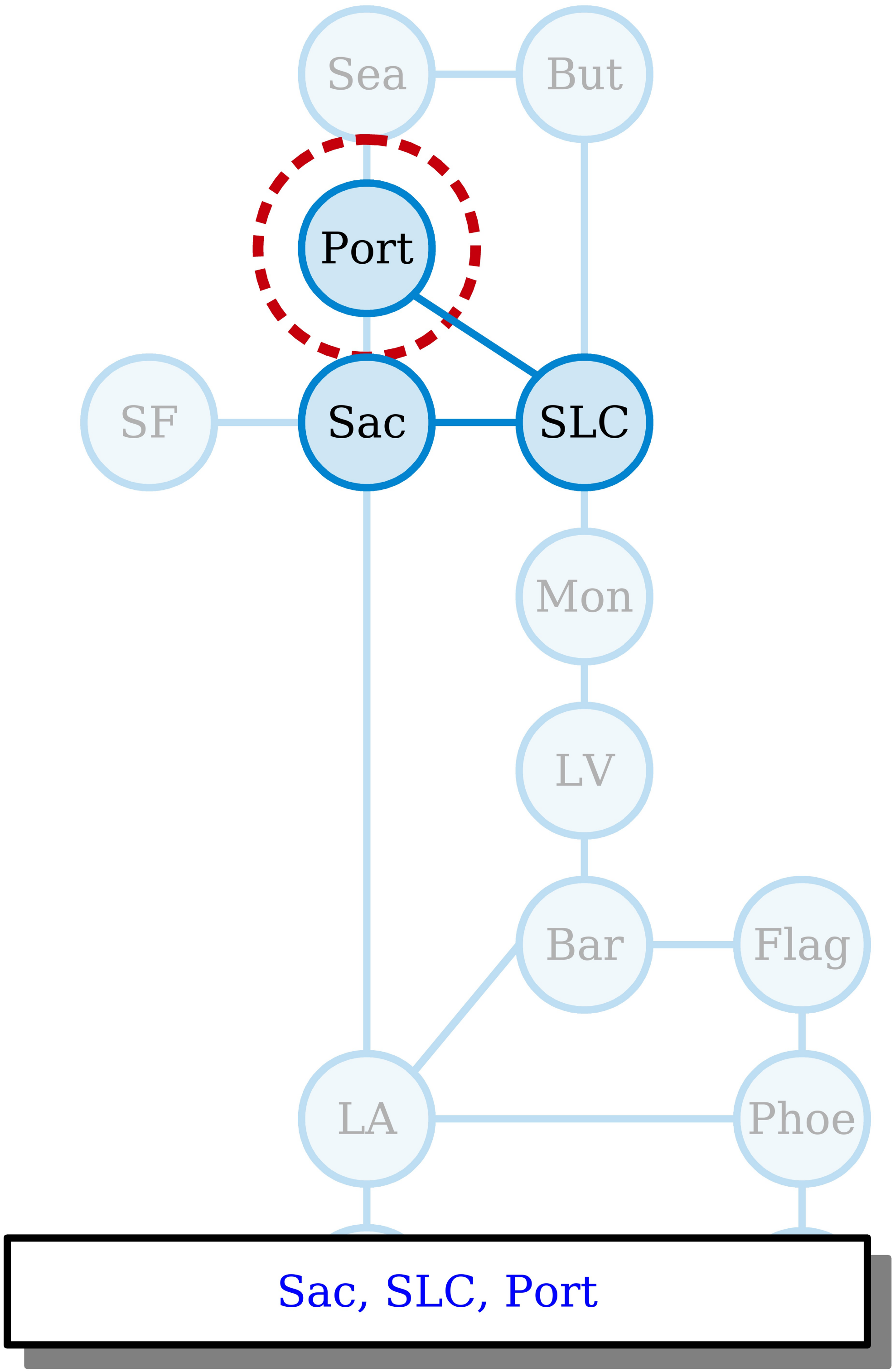

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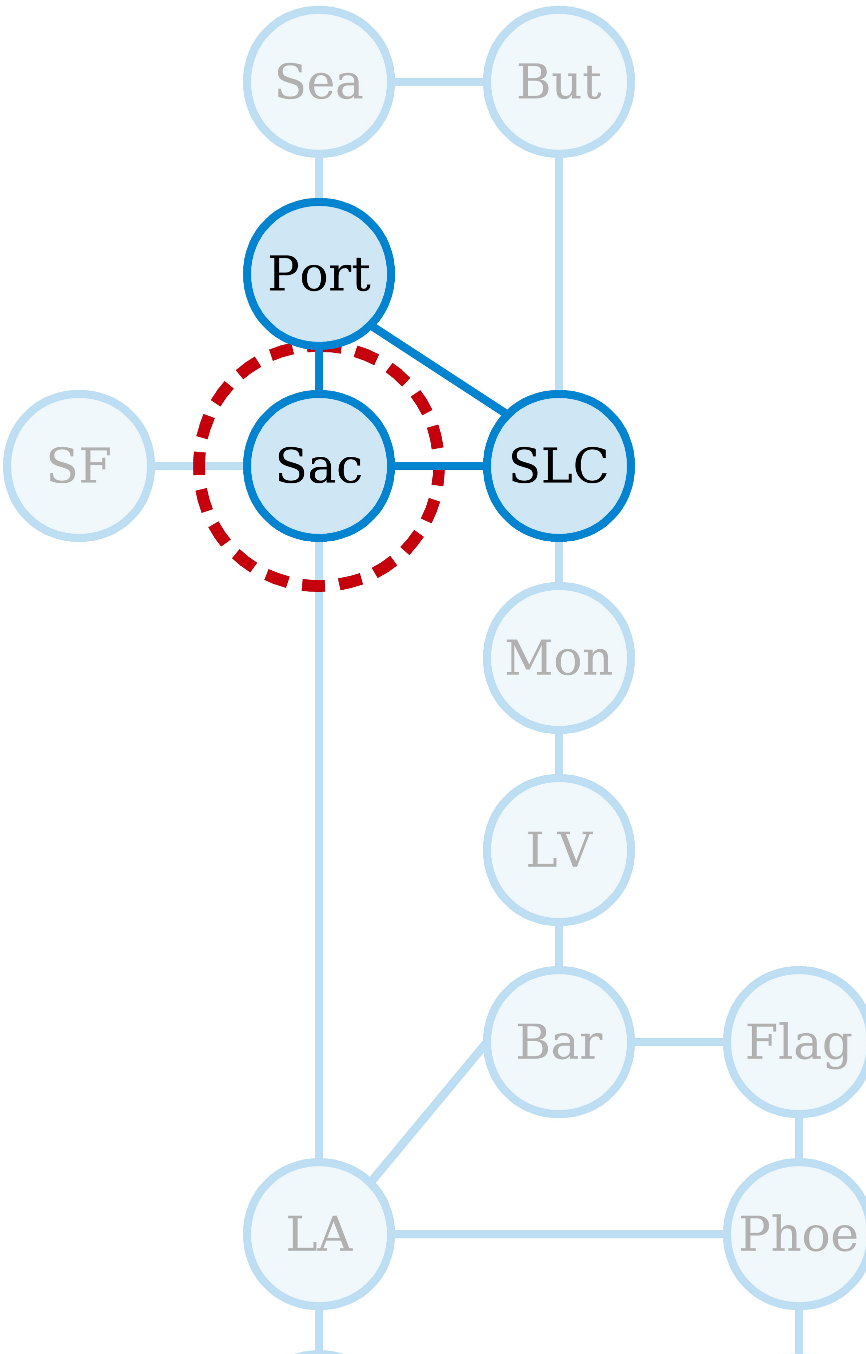

Sac, SLC, Port, Sac

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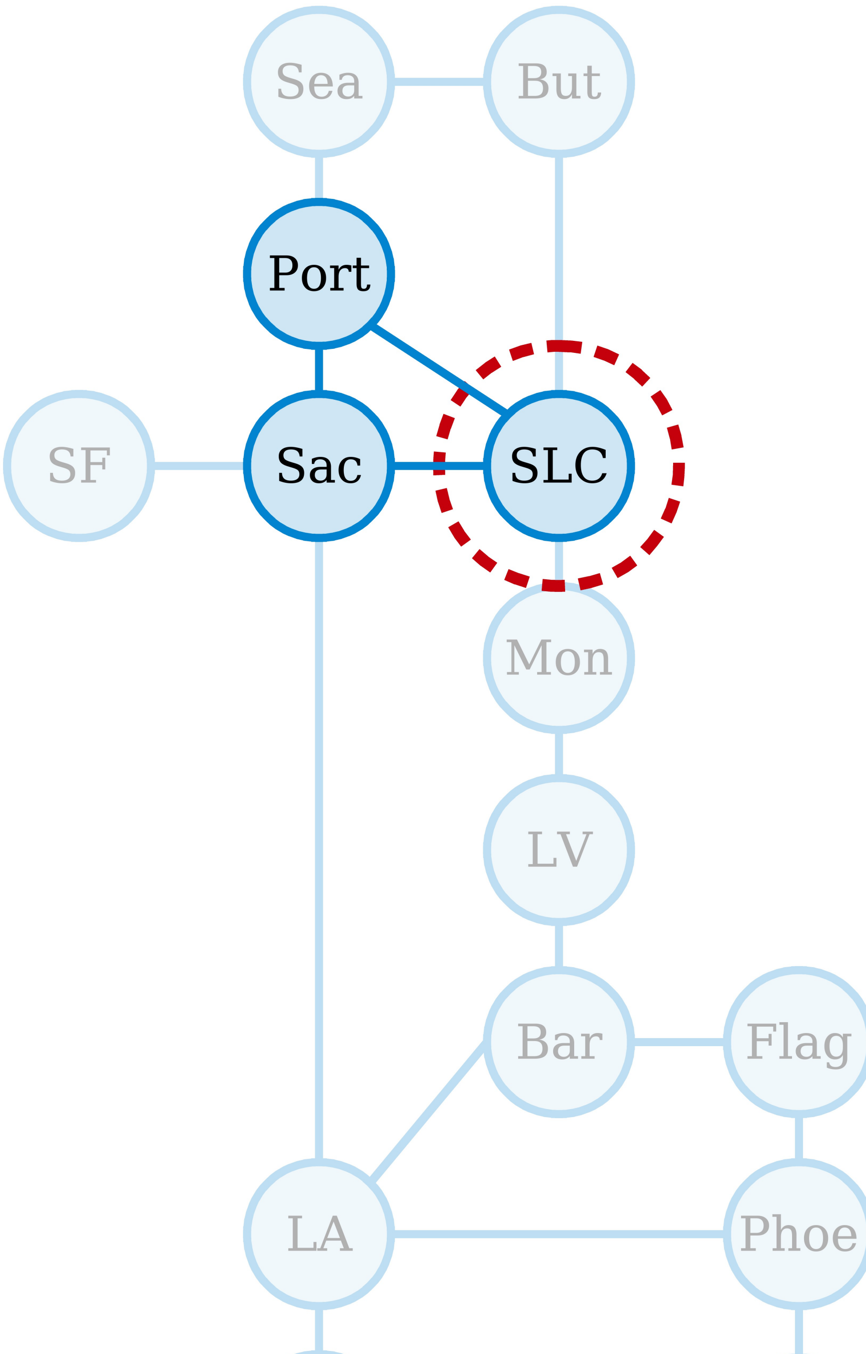

Sac, SLC, Port, Sac, SLC

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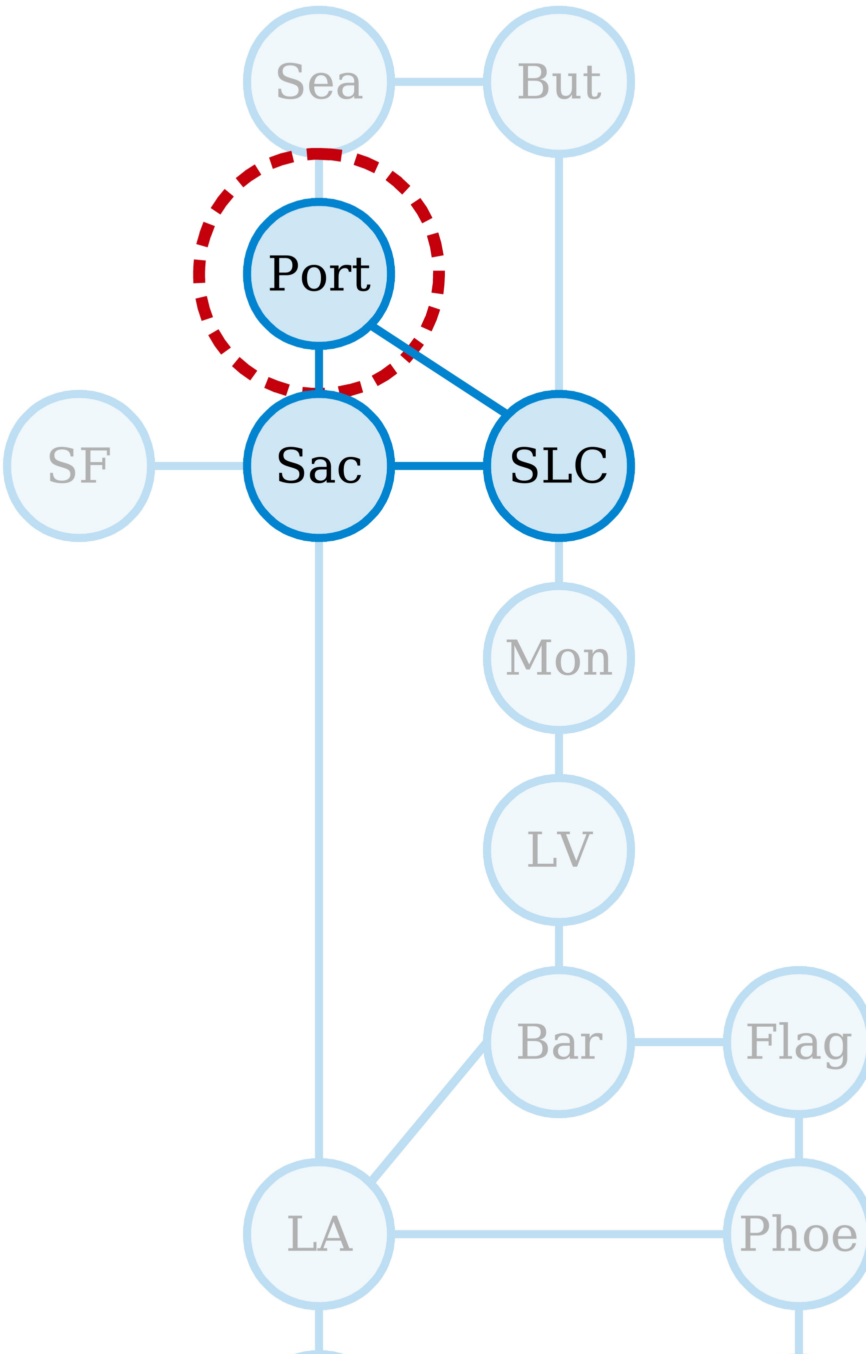

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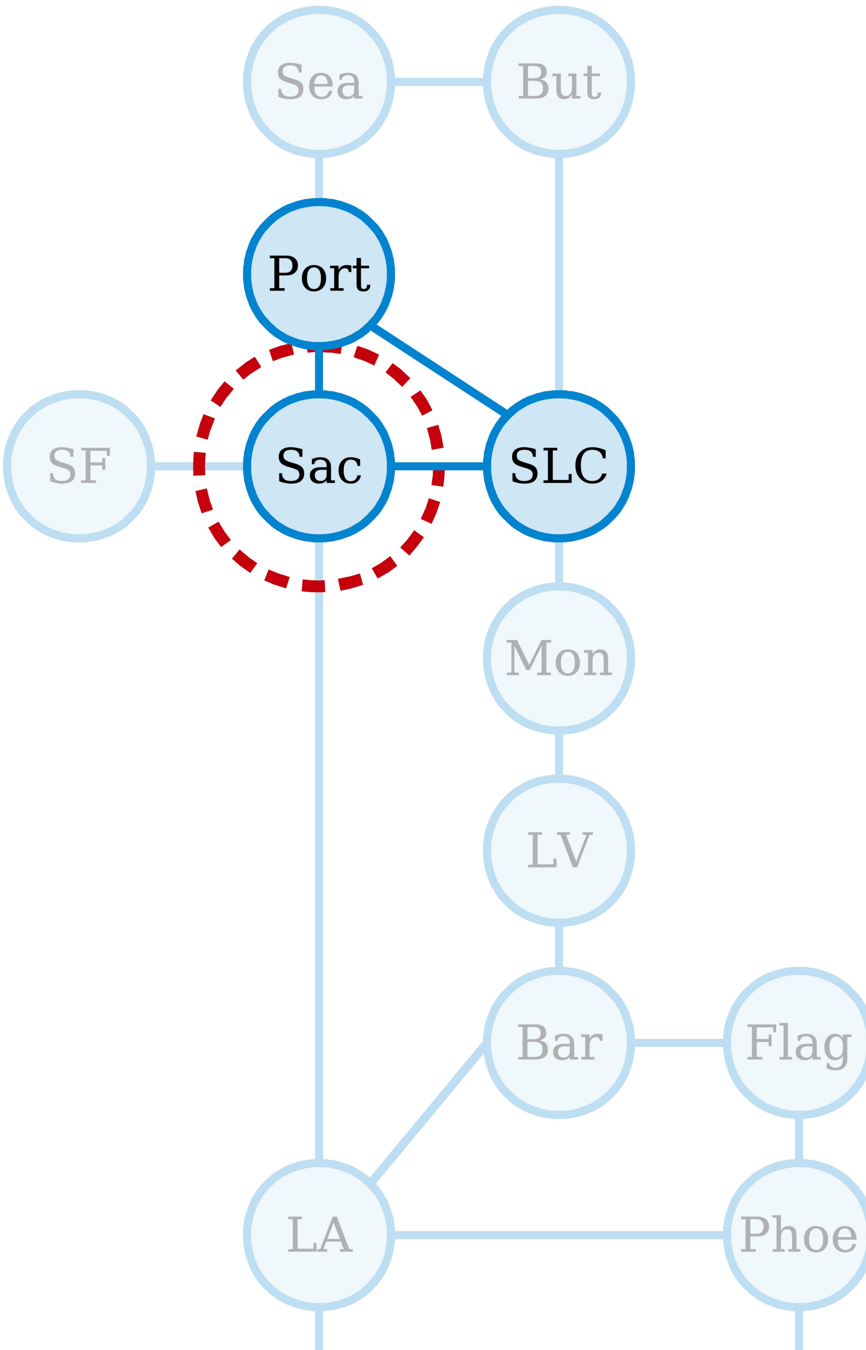

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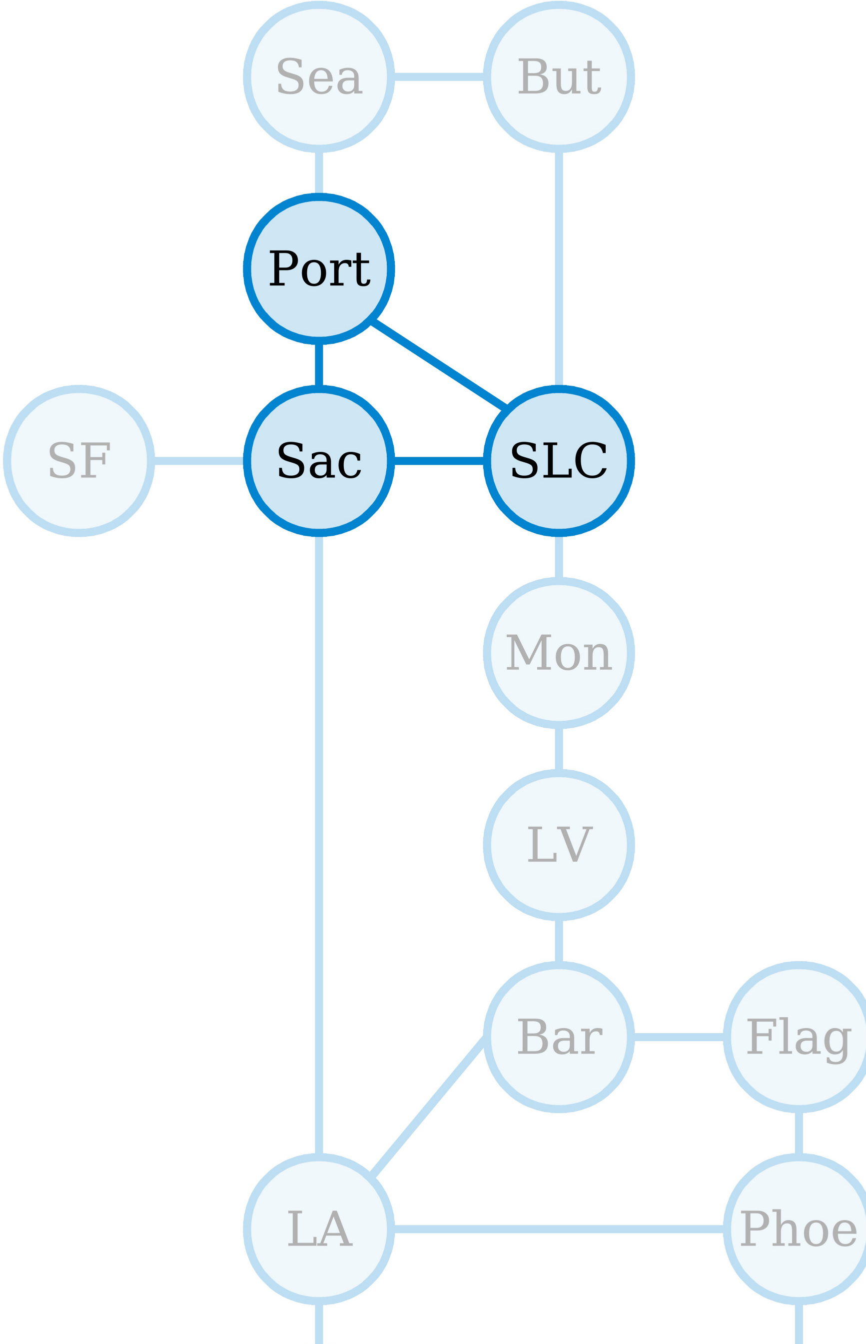

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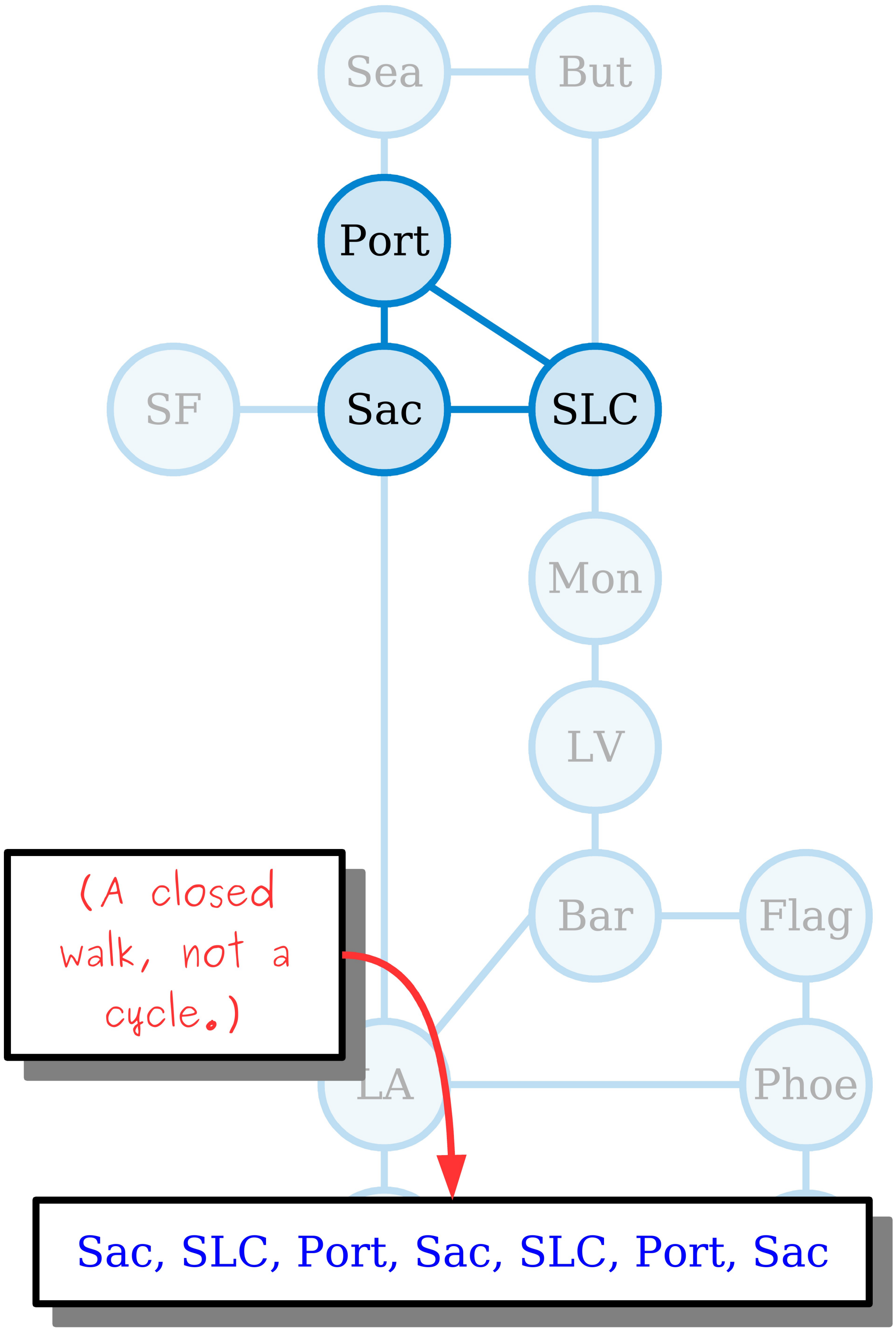

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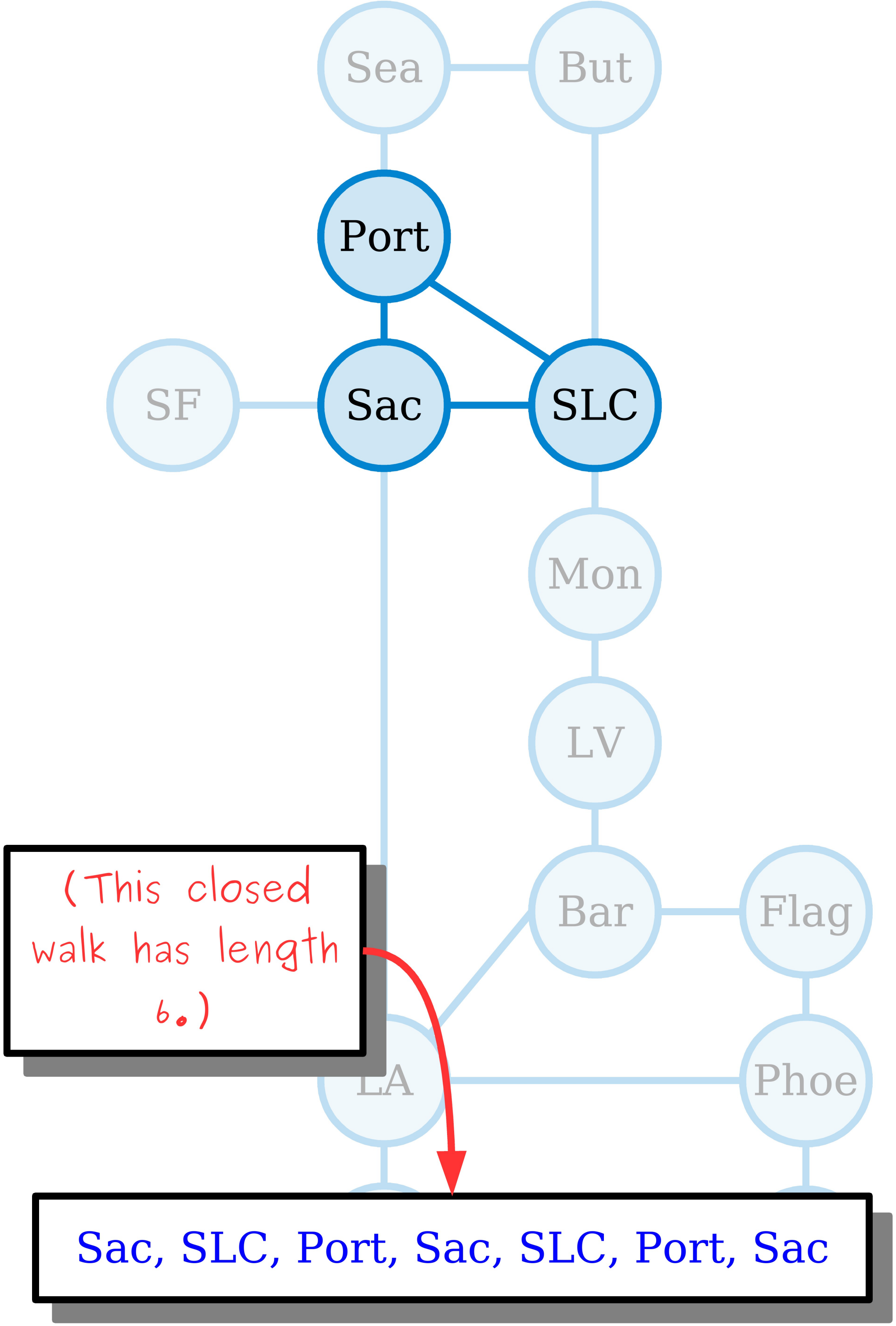

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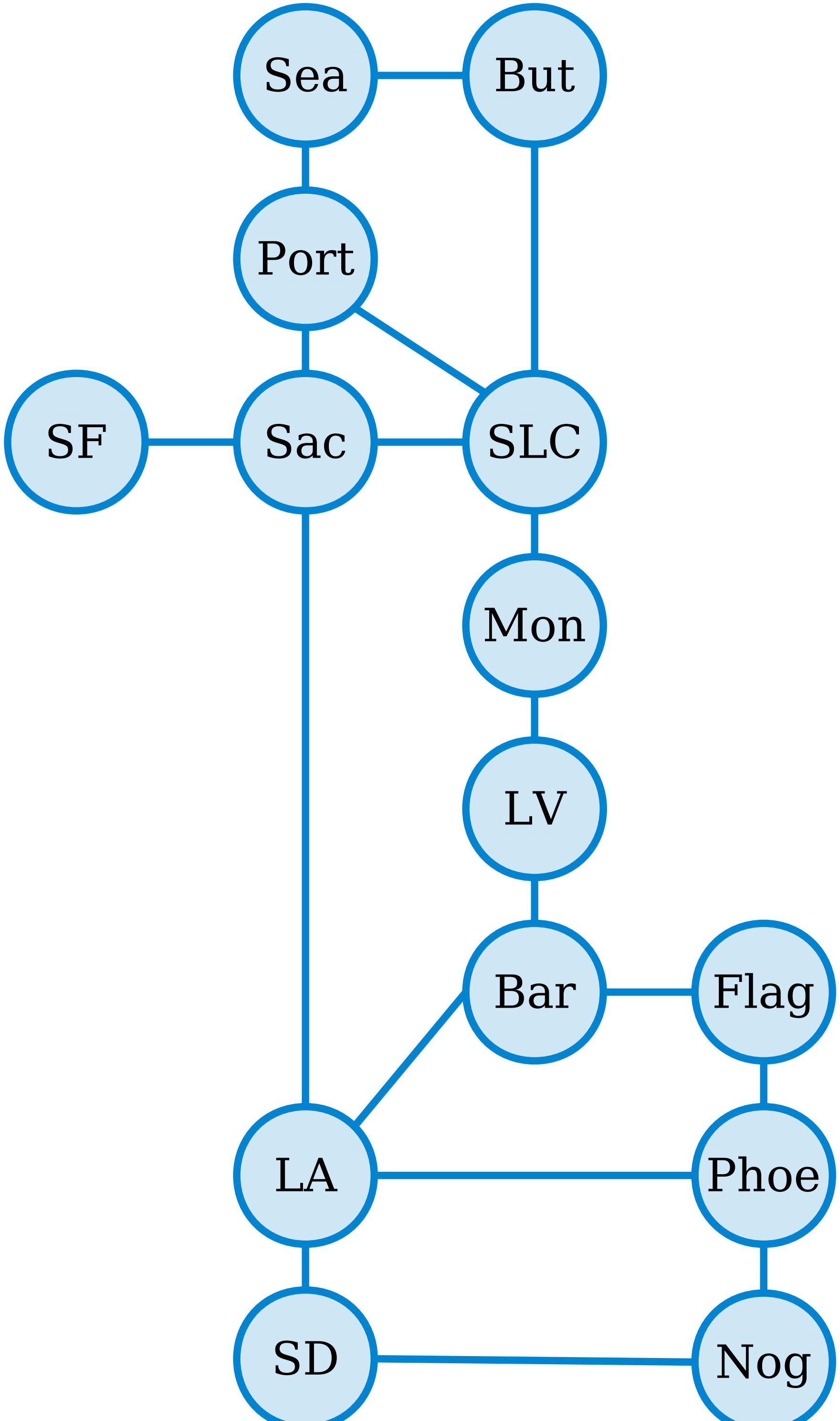

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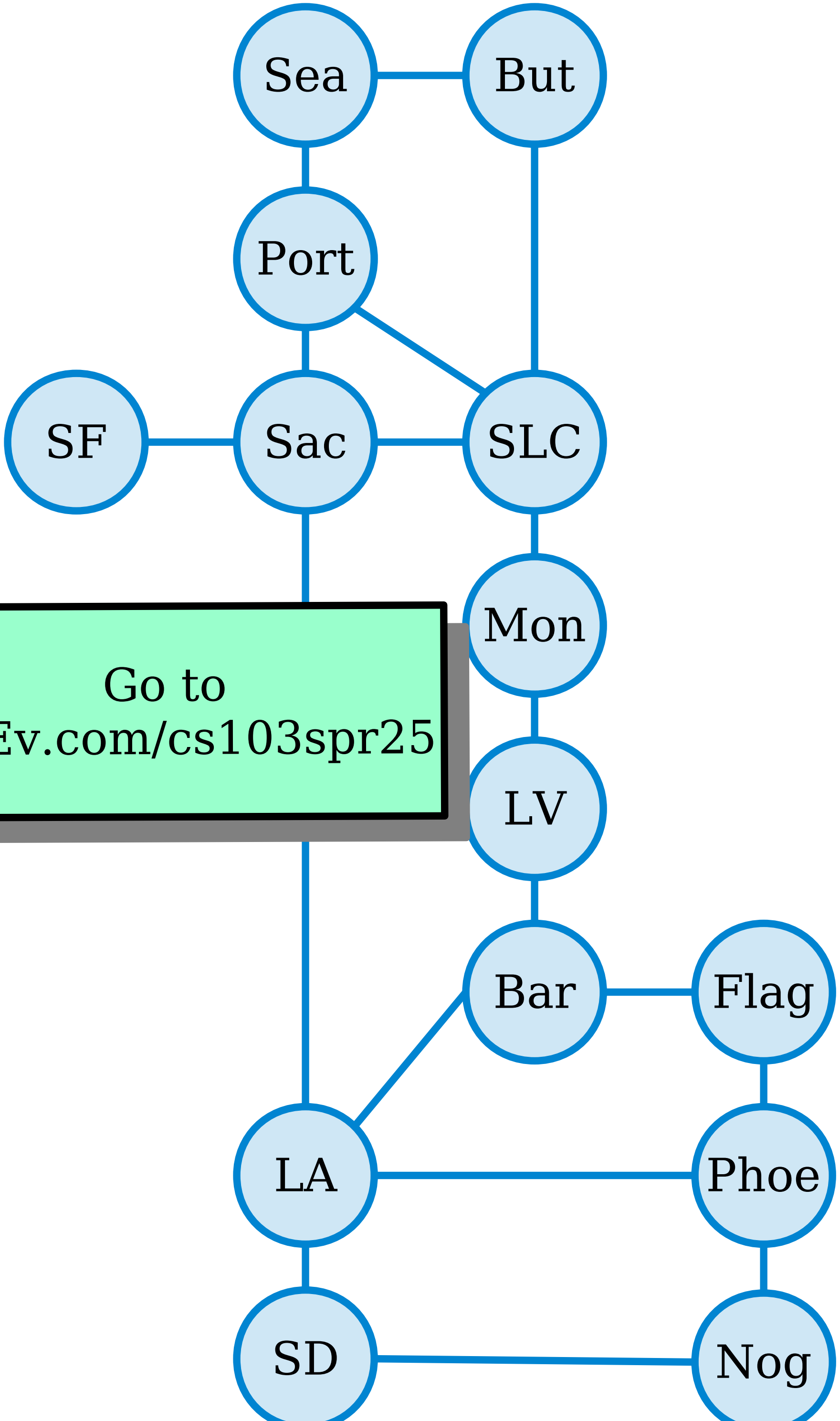

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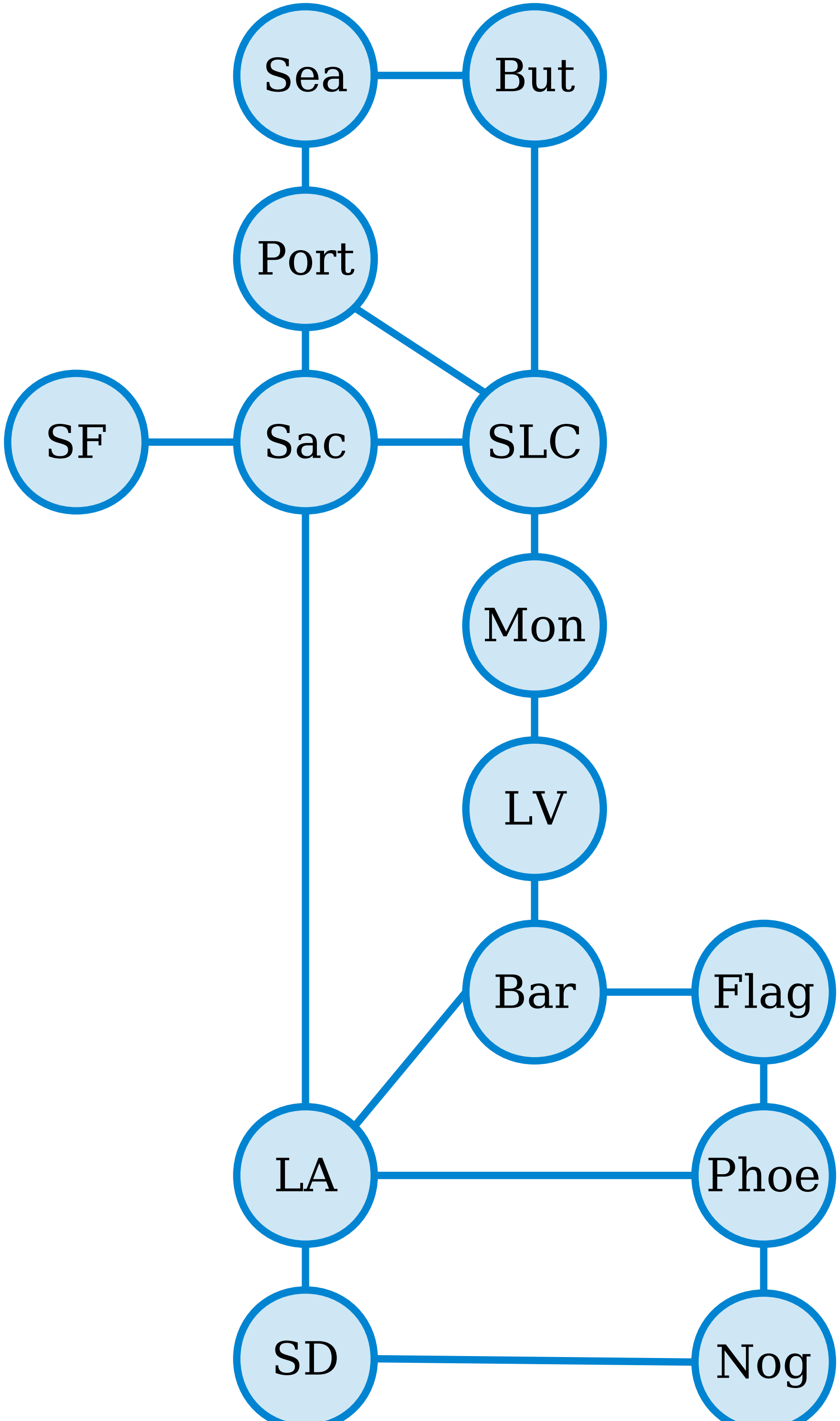

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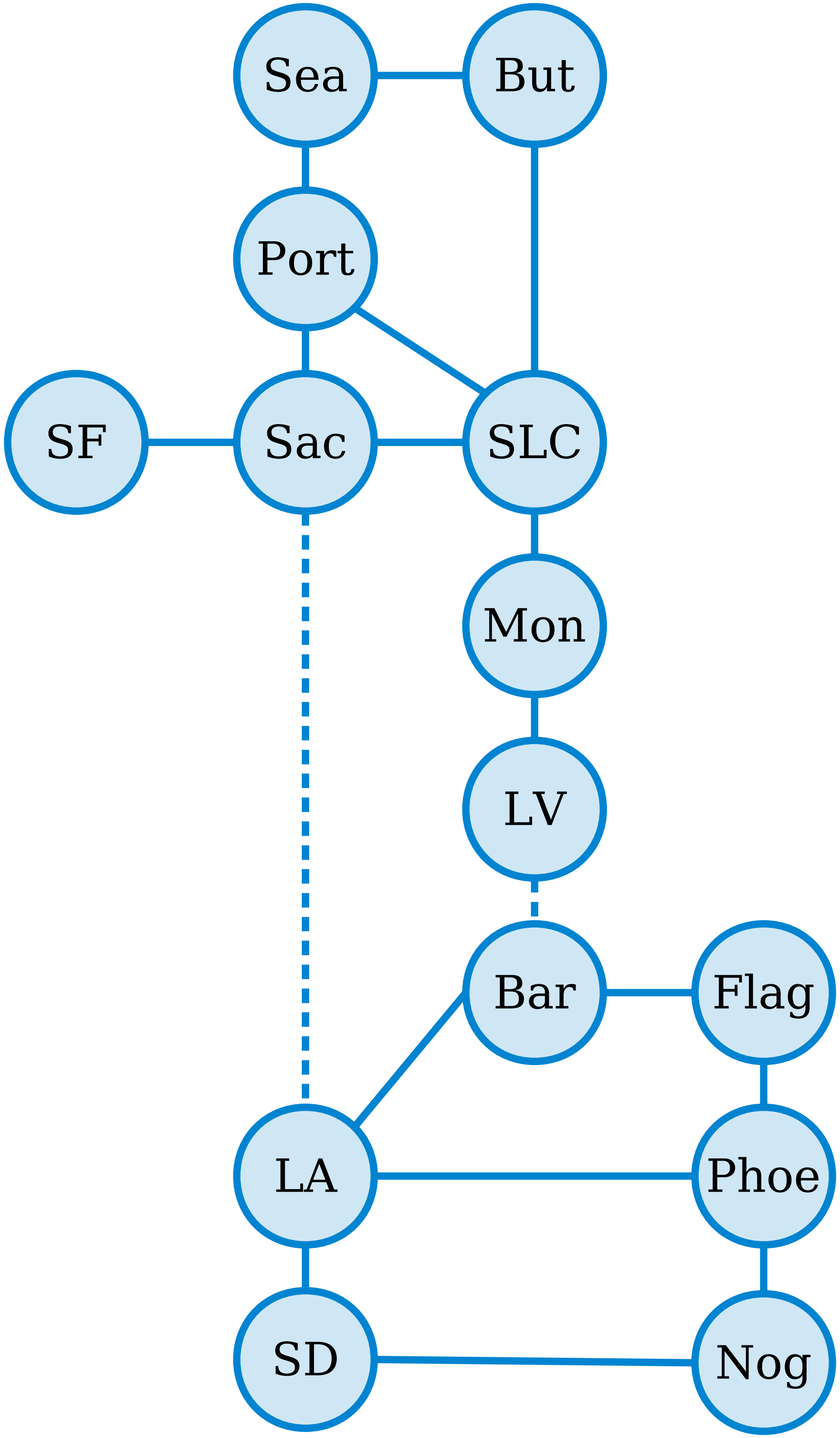

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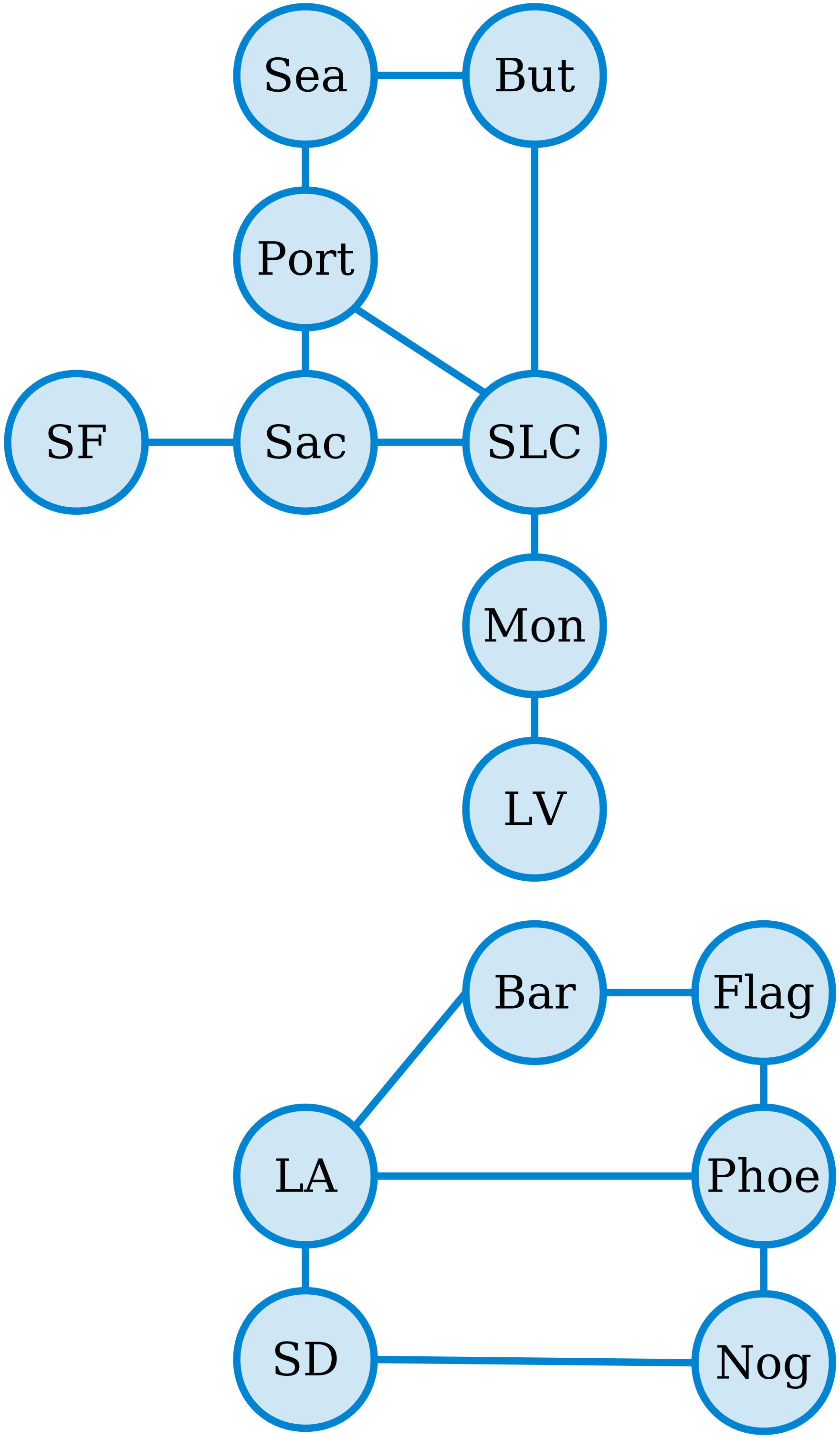

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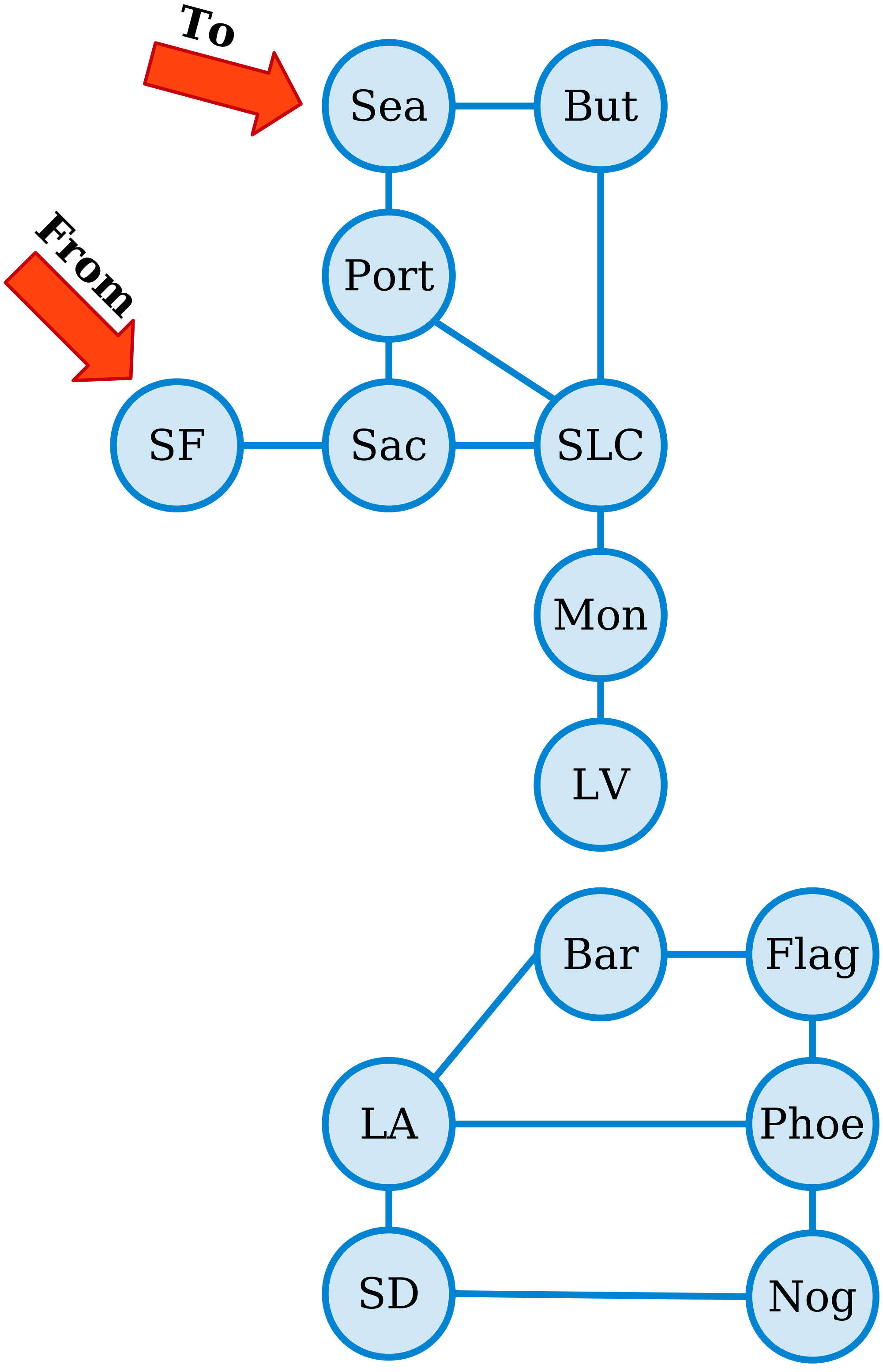

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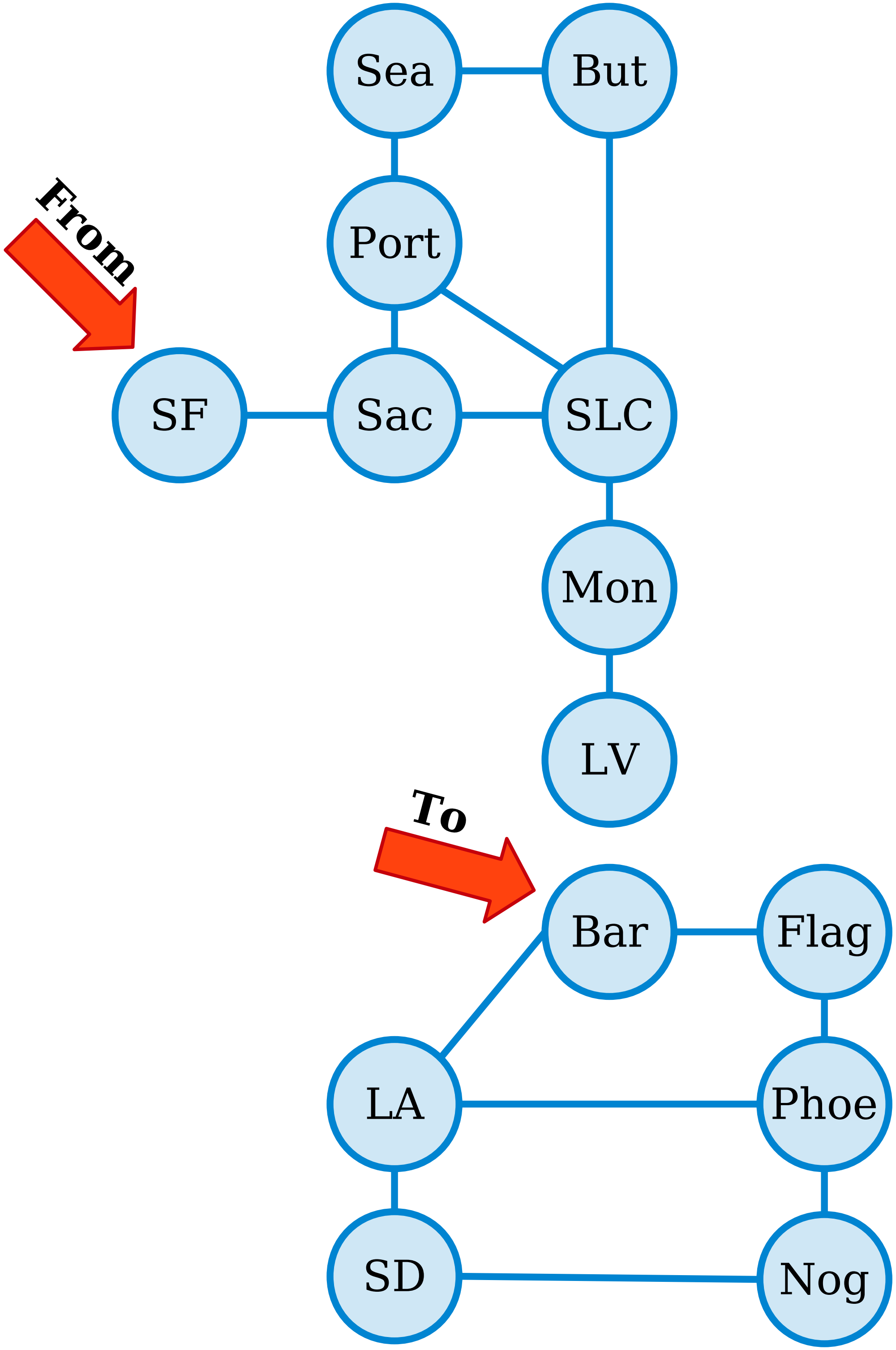

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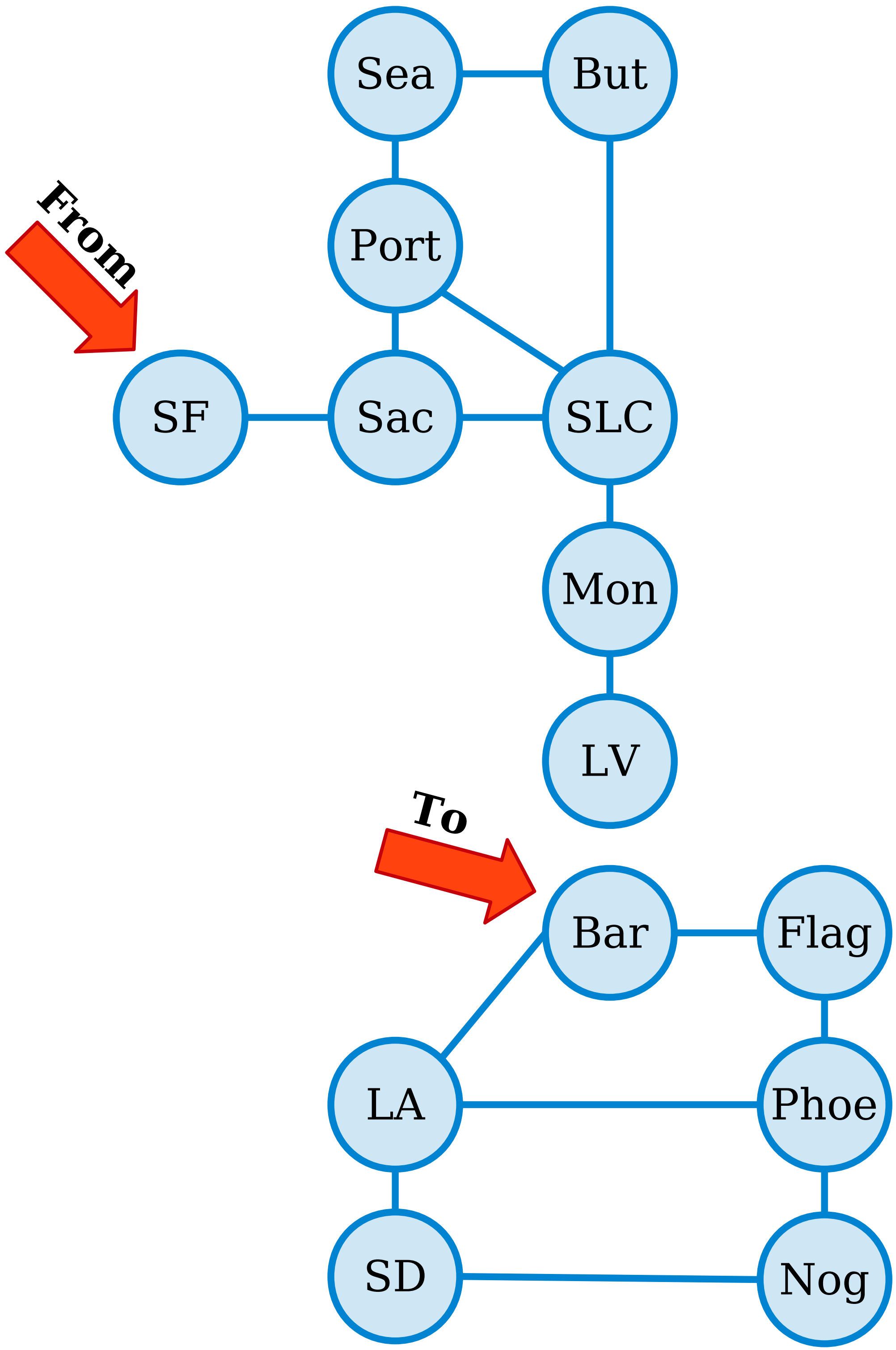

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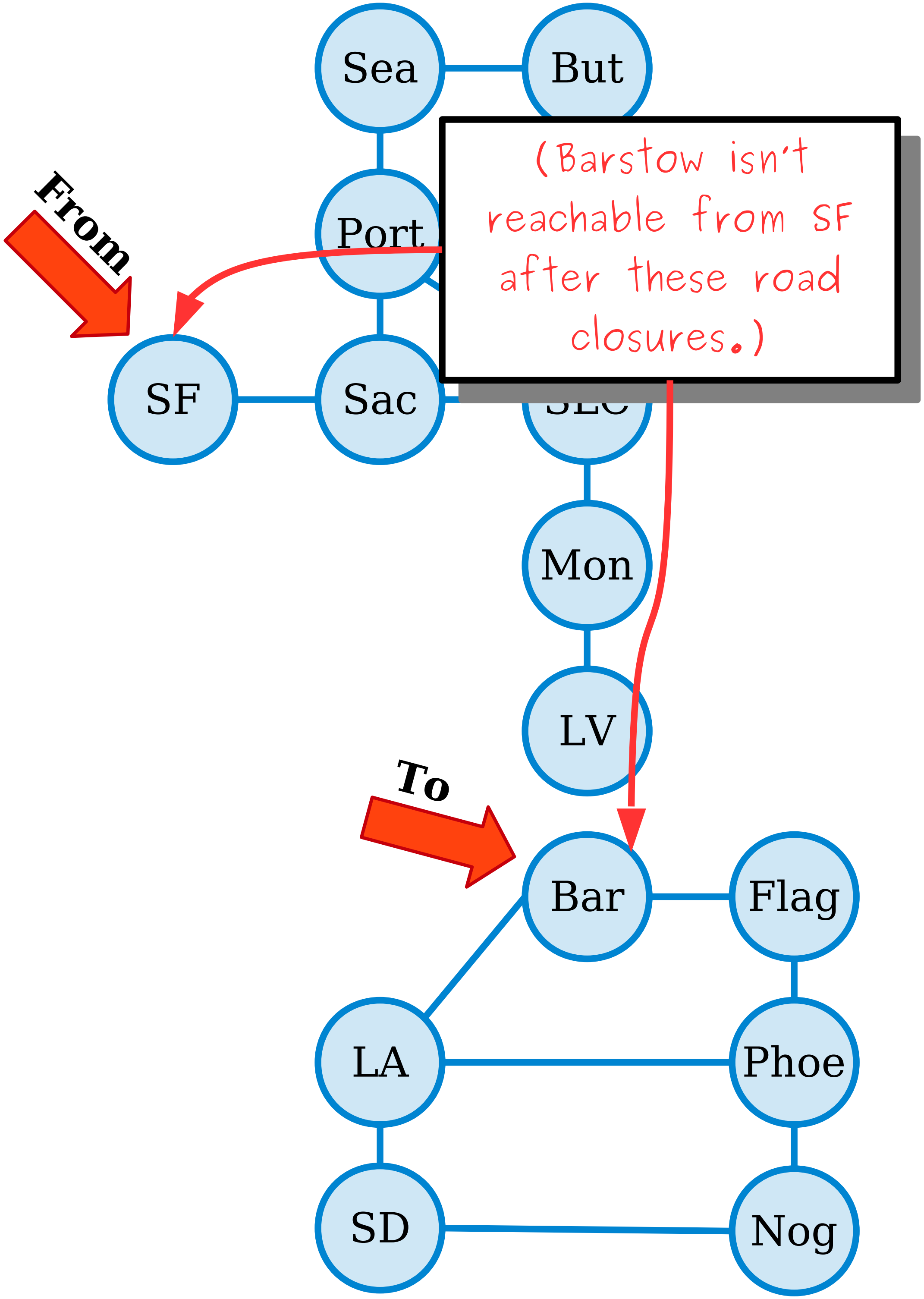

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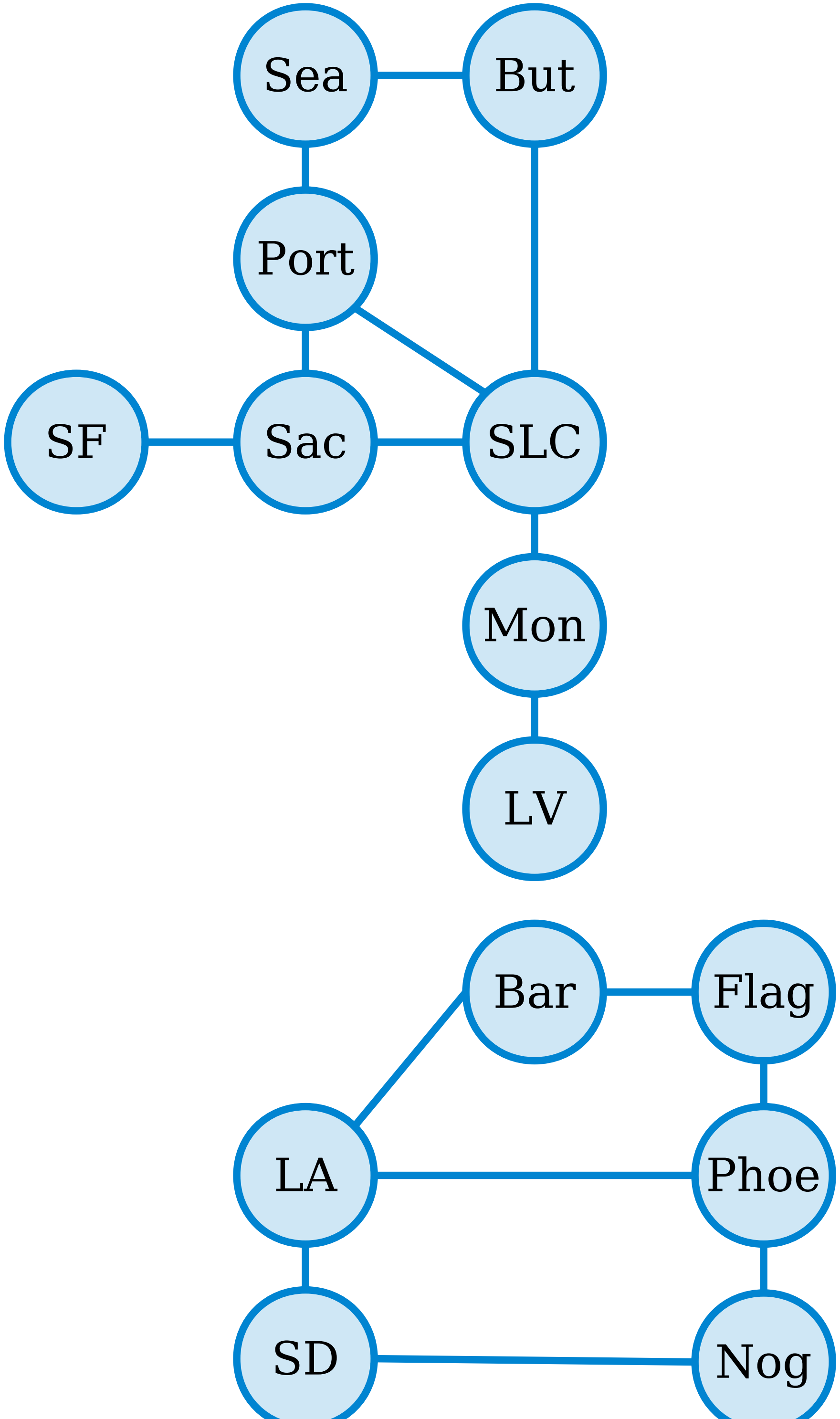

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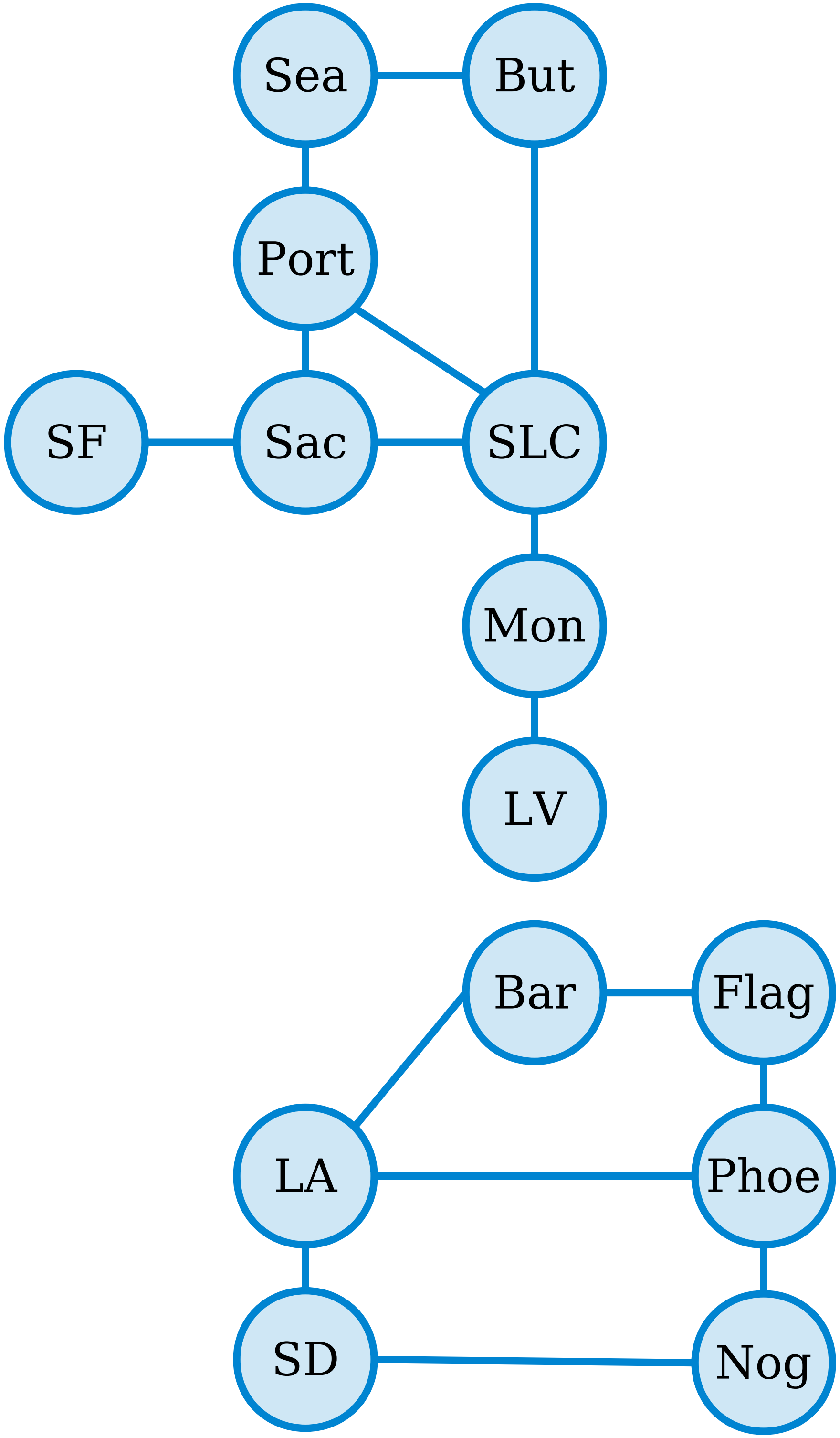

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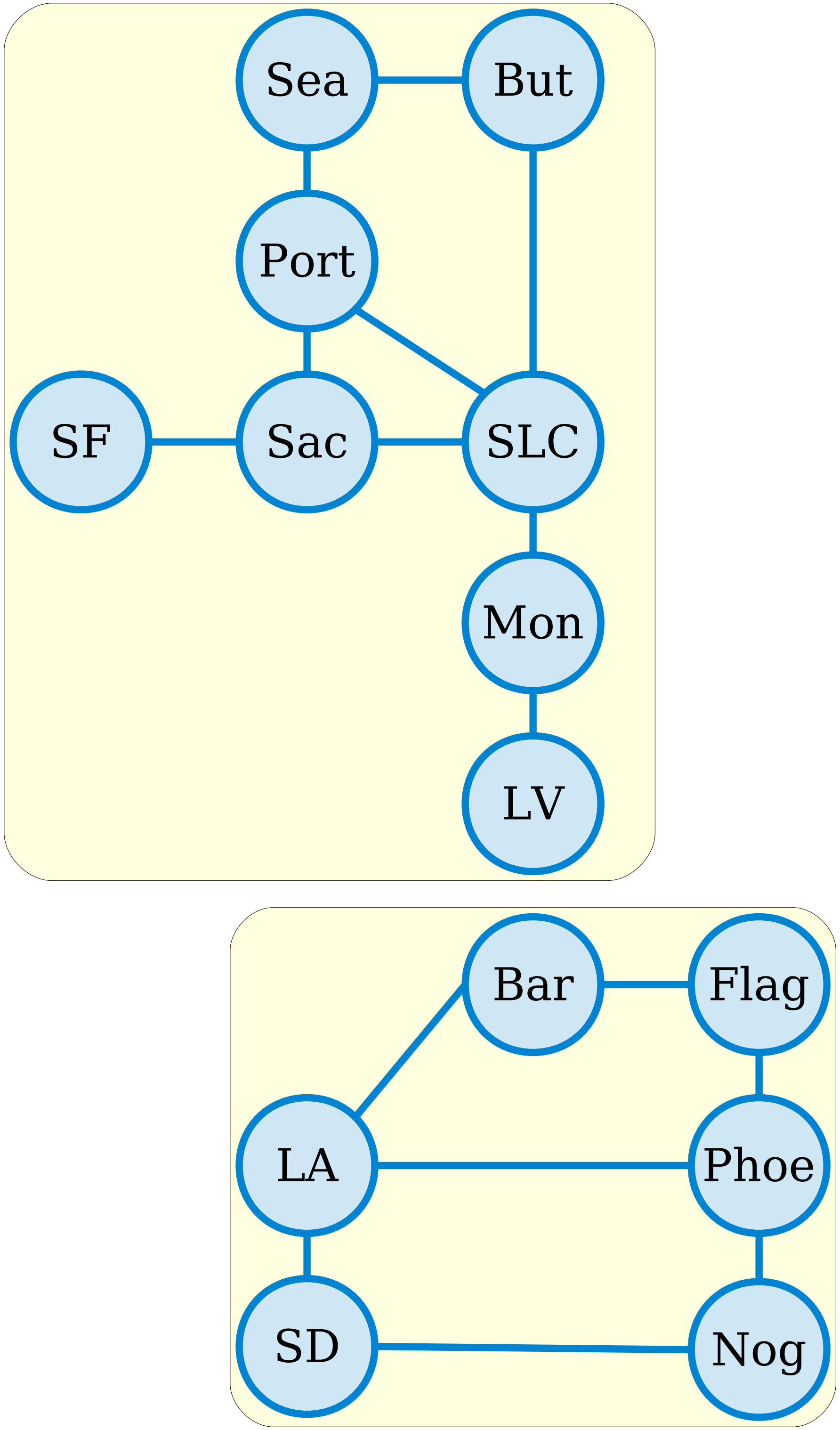
(This graph is not connected.)



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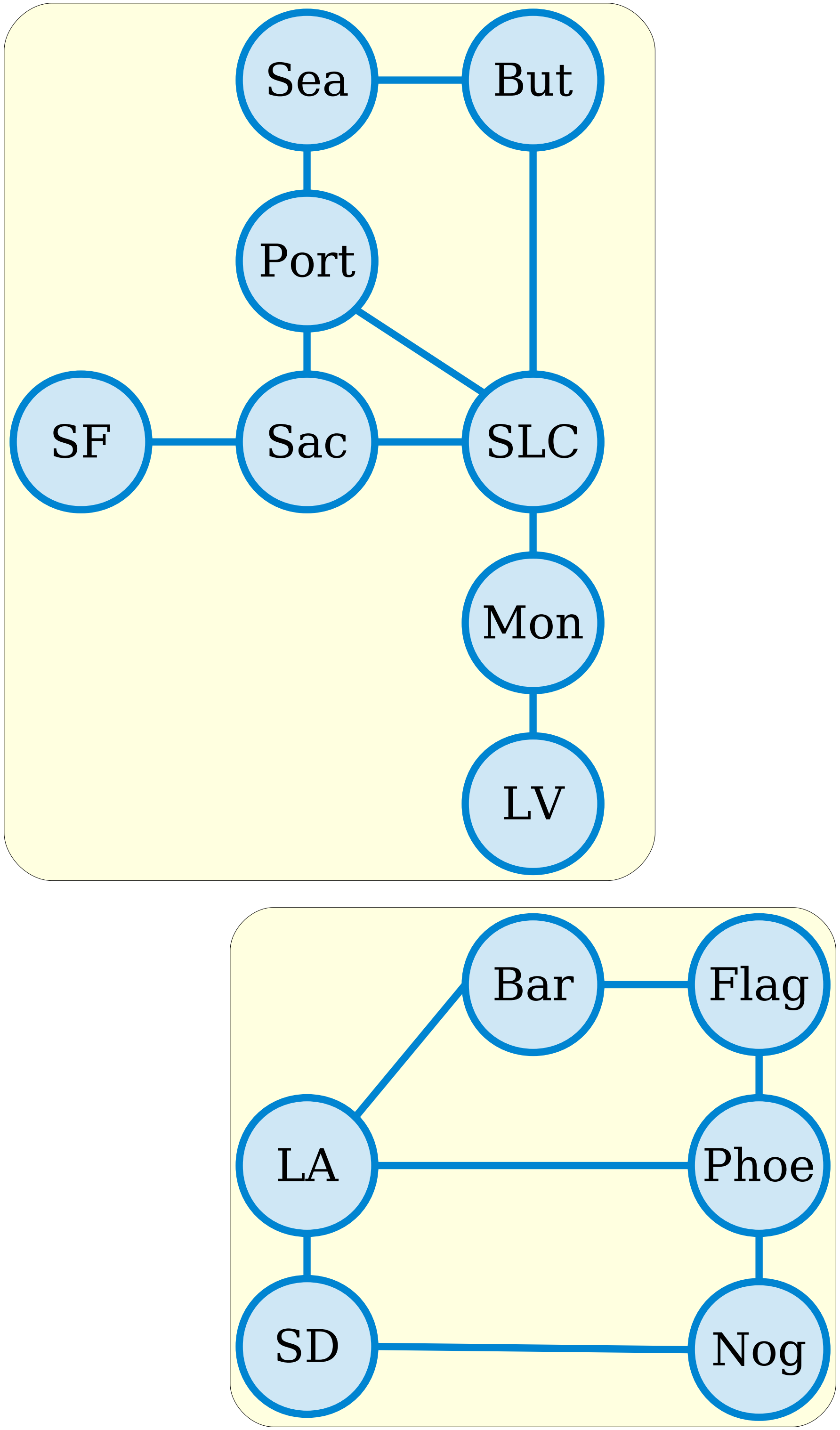

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A ***connected component*** (or ***CC***) of G is a set consisting of a node and every node reachable from it.

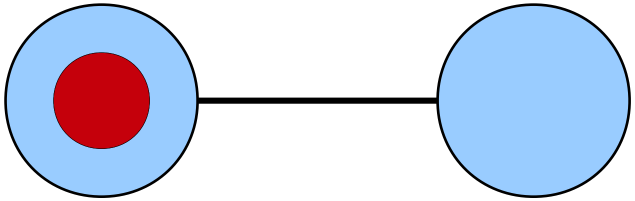
Fun Facts

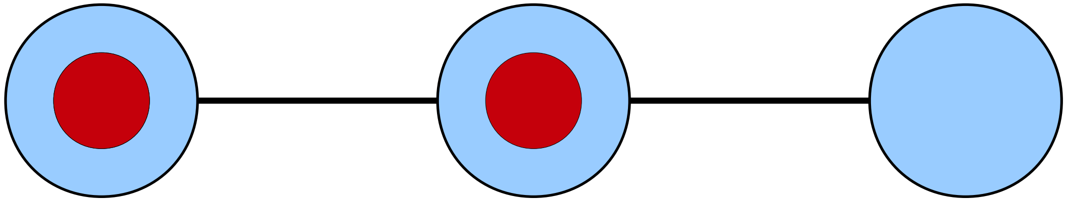
- Here's a collection of useful facts about graphs that you can take as a given.
  - ***Theorem:*** If $G = (V, E)$ is a graph and $u, v \in V$, then there is a path from u to v if and only if there's a walk from u to v.
  - ***Theorem:*** If G is a graph and C is a cycle in G, then C's length is at least three and C contains at least three nodes.
  - ***Theorem:*** If $G = (V, E)$ is a graph, then every node in V belongs to exactly one connected component of G.
  - ***Theorem:*** If $G = (V, E)$ is a graph, then G is not connected if and only if G has two or more connected components.
- Looking for more practice working with formal definitions? Prove these results!

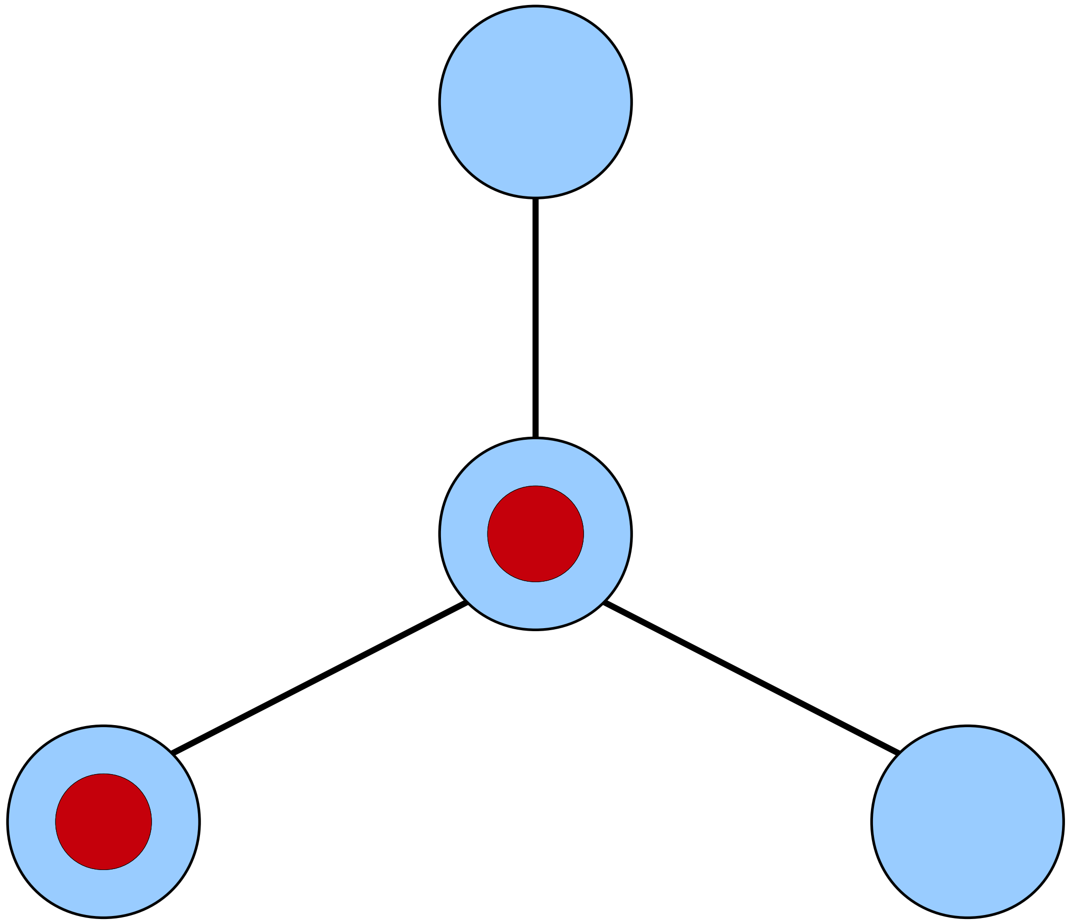
Application: ***Local Area Networks***

The Internet and LANs

- The internet consists of several separate ***local area networks*** (***LANs***) that are "internetworked" together.
- Local area networks cover small areas – a single hallway in a dorm, an office building, a college campus, etc.
- The internet then links those smaller LANs into one giant network where everyone can talk to everyone.
- ***Focus for today:*** How do messages flow through a LAN?

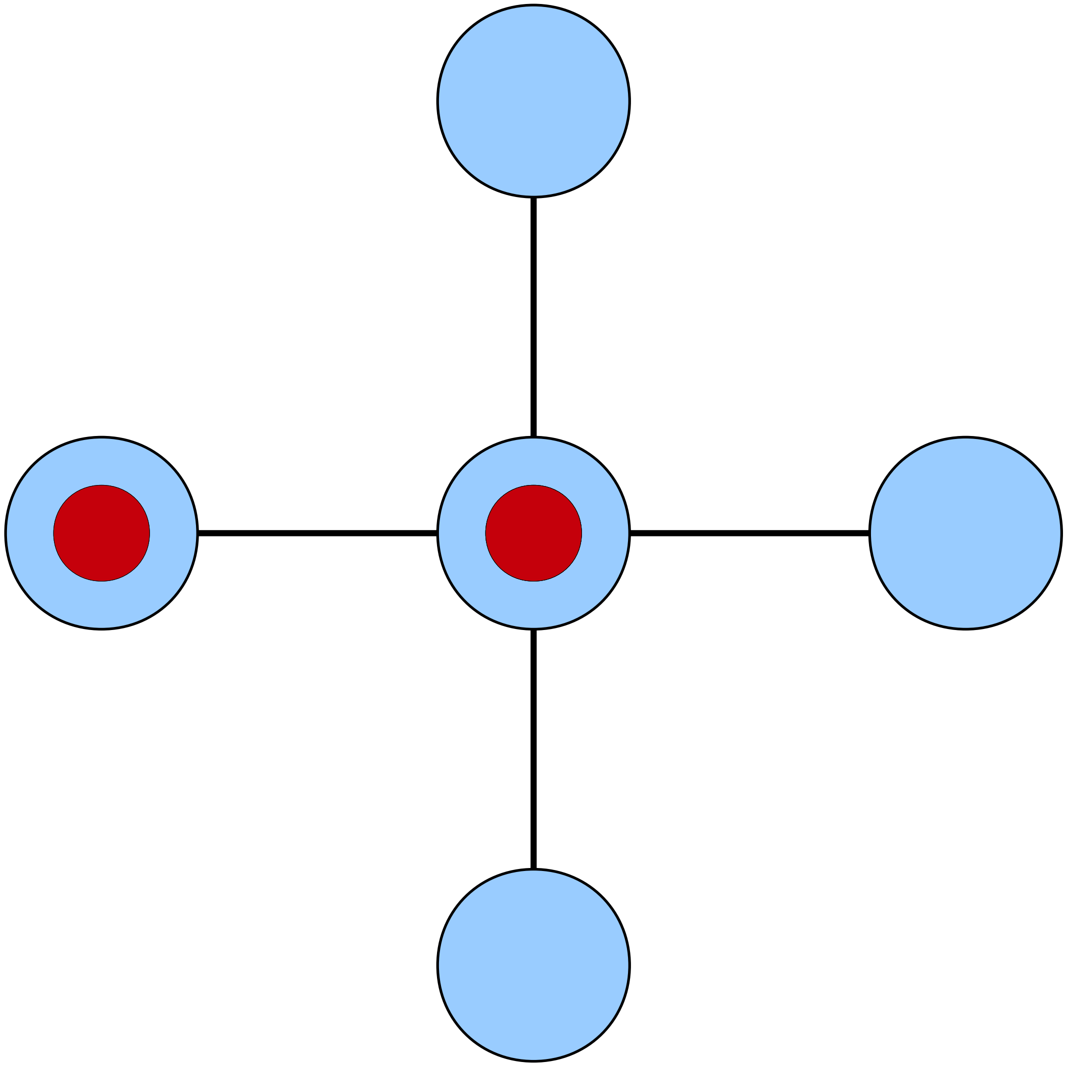


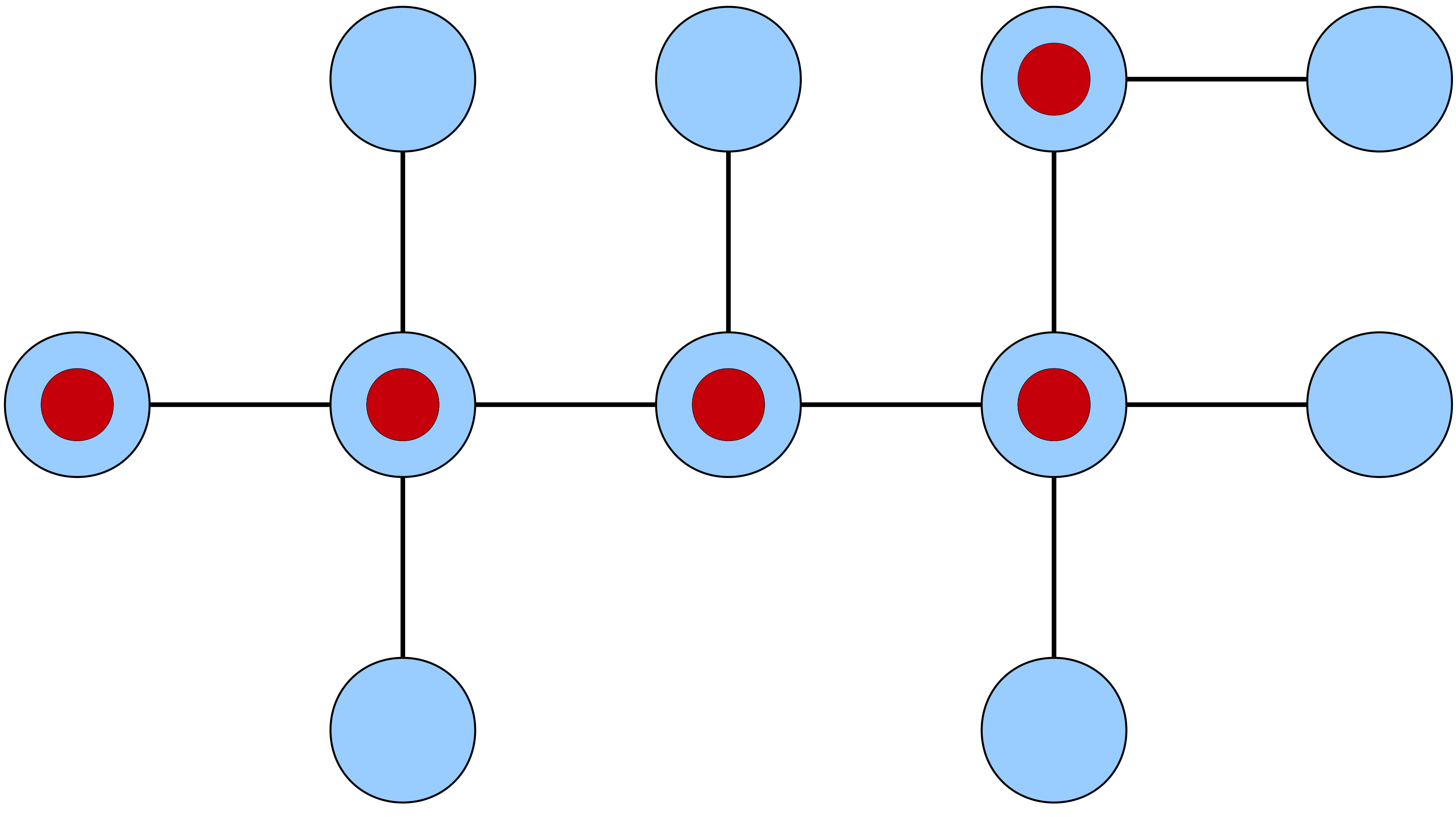




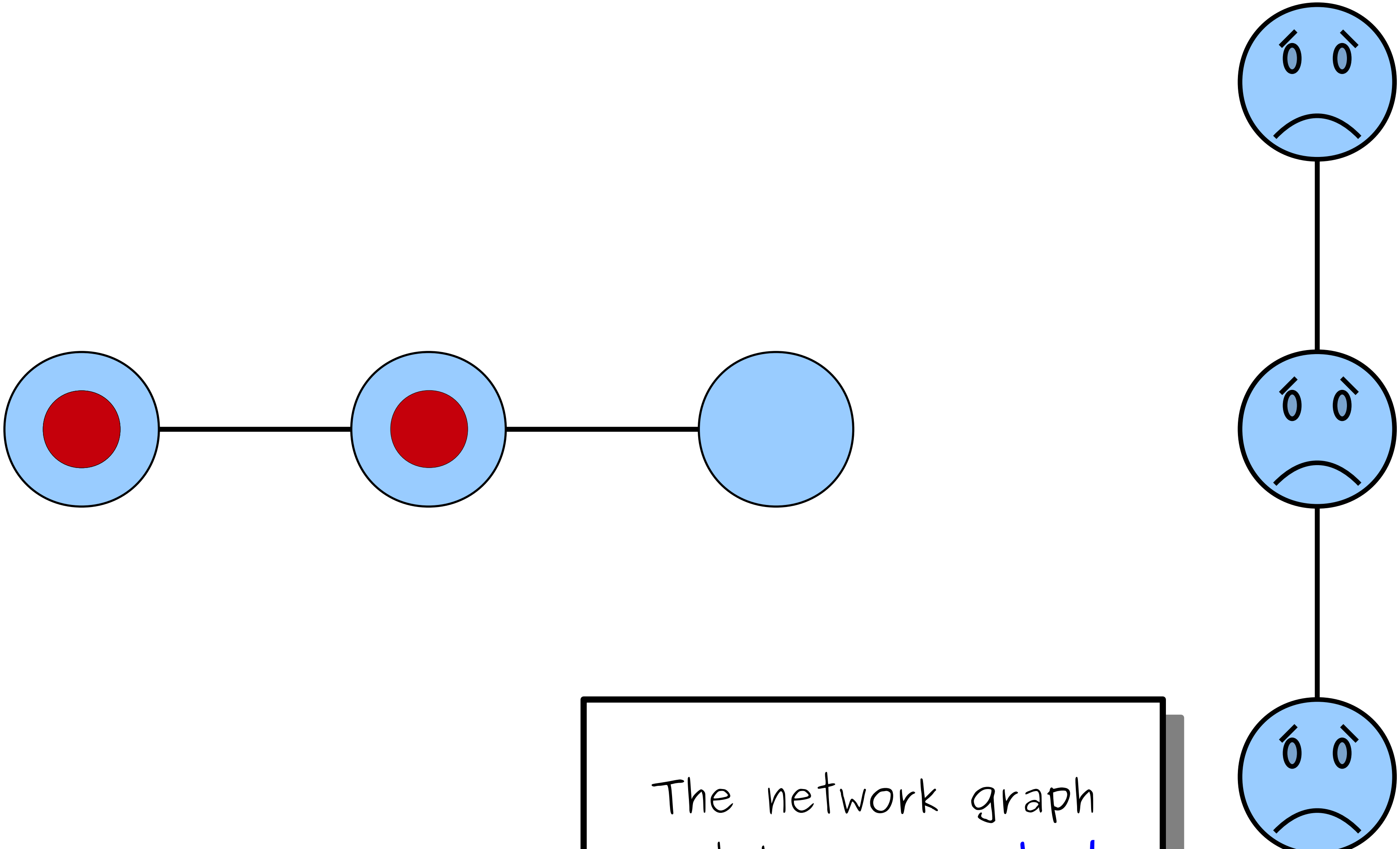
Message Movement

- When a computer receives a message, it repeats that message on all its links except the one it received the message on.
- The computers don't inspect the message contents or try to be clever – it's purely "came in on link *X*, goes out on all links but *X*."

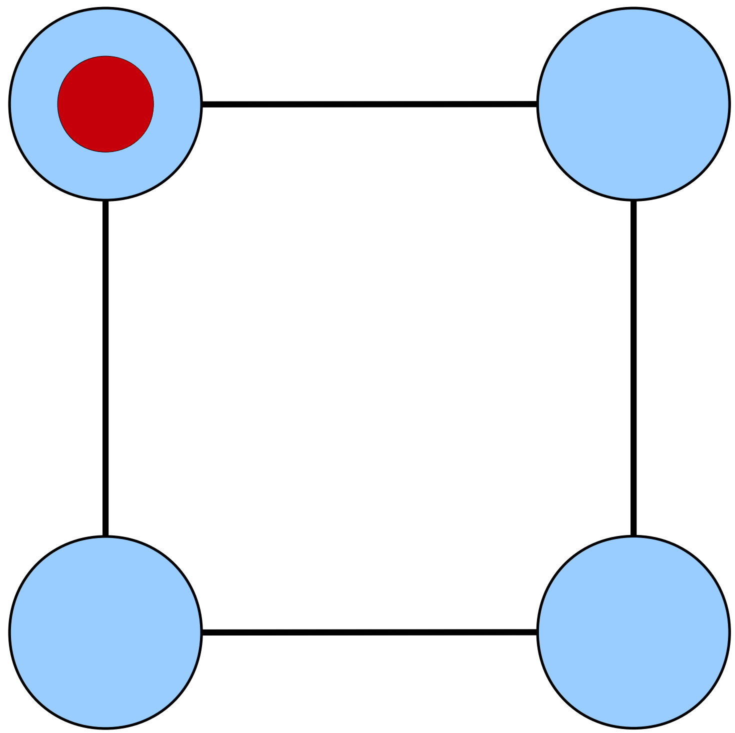




Two Pitfalls



The network graph must be **connected**.



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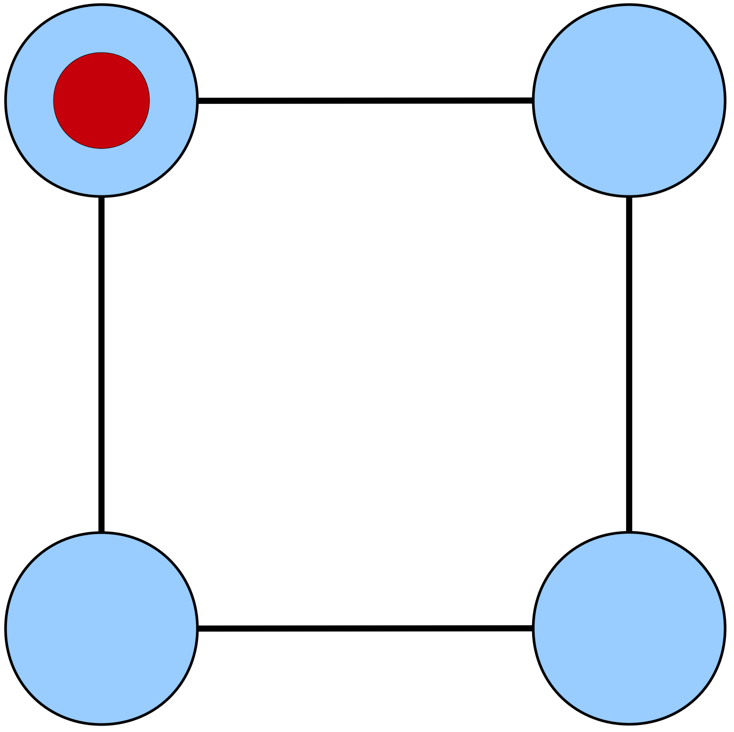

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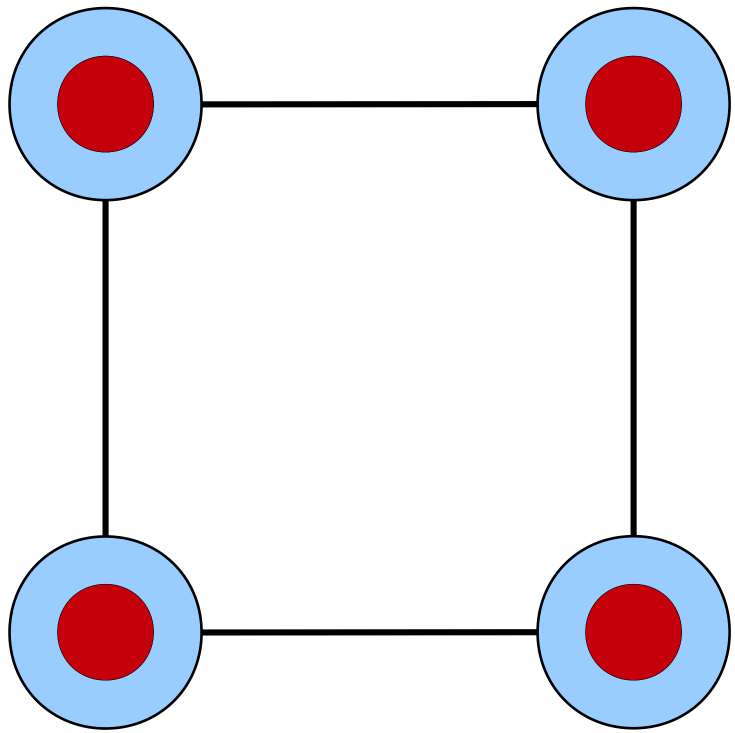
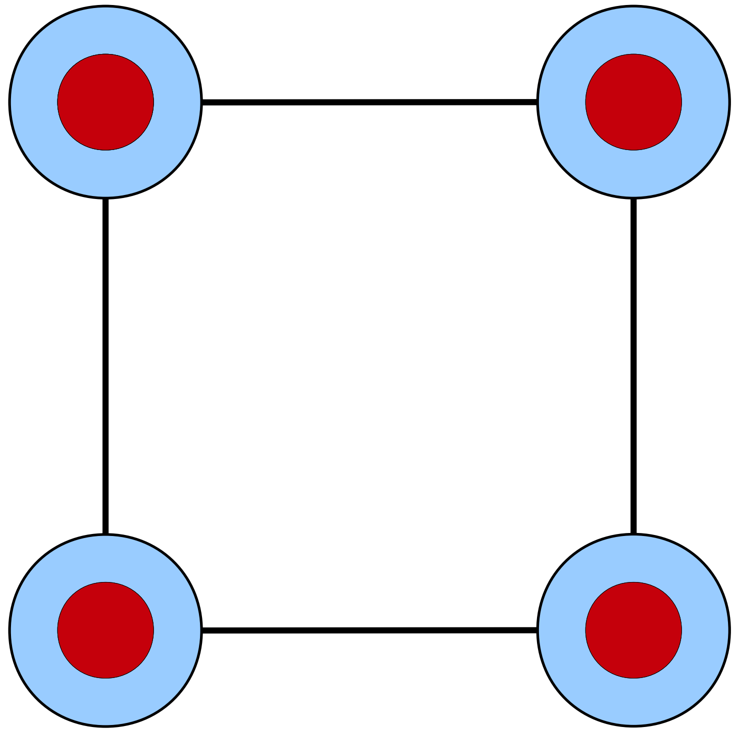
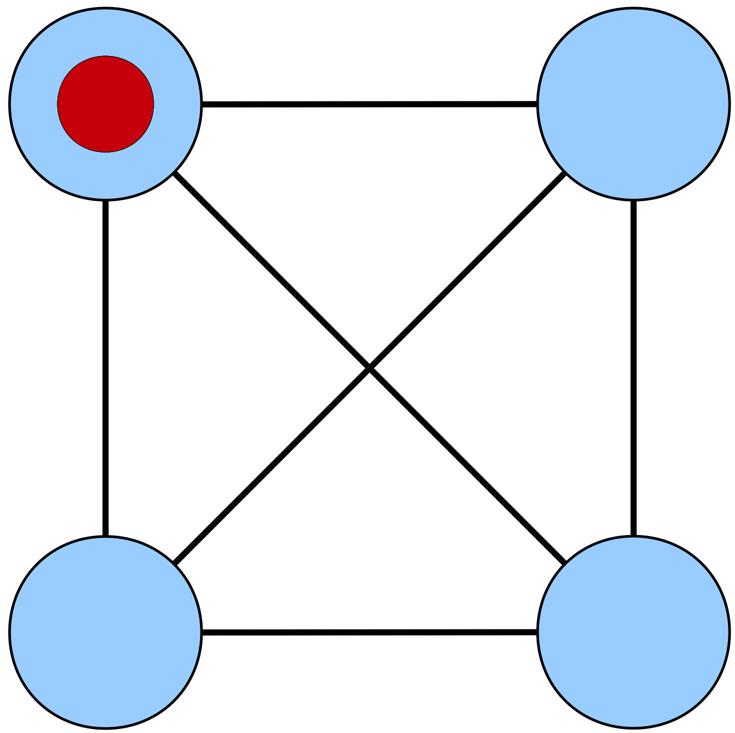
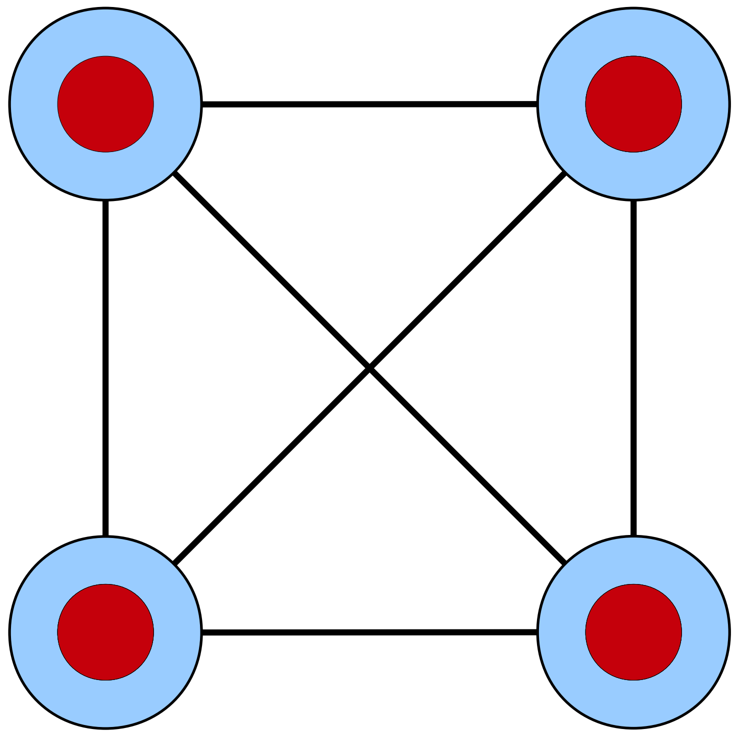
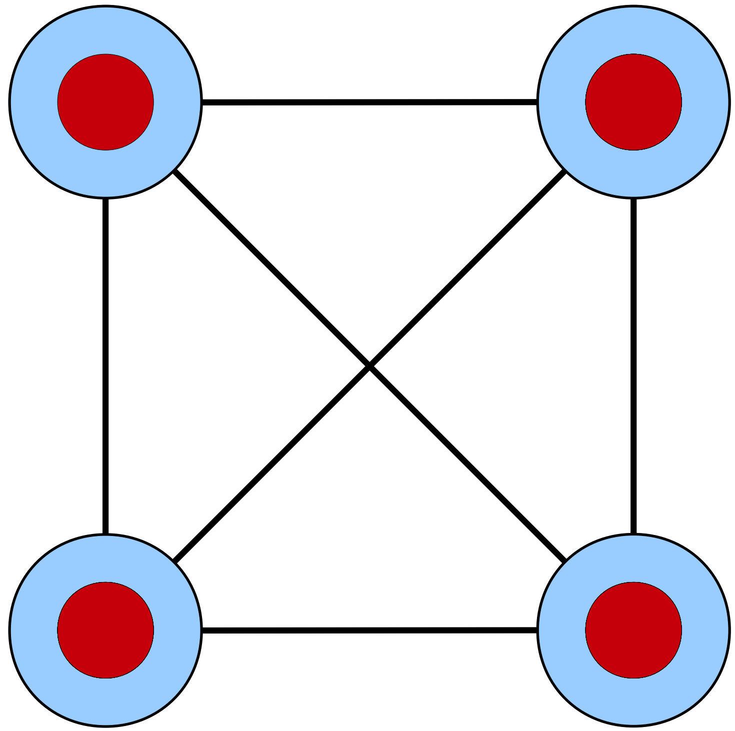




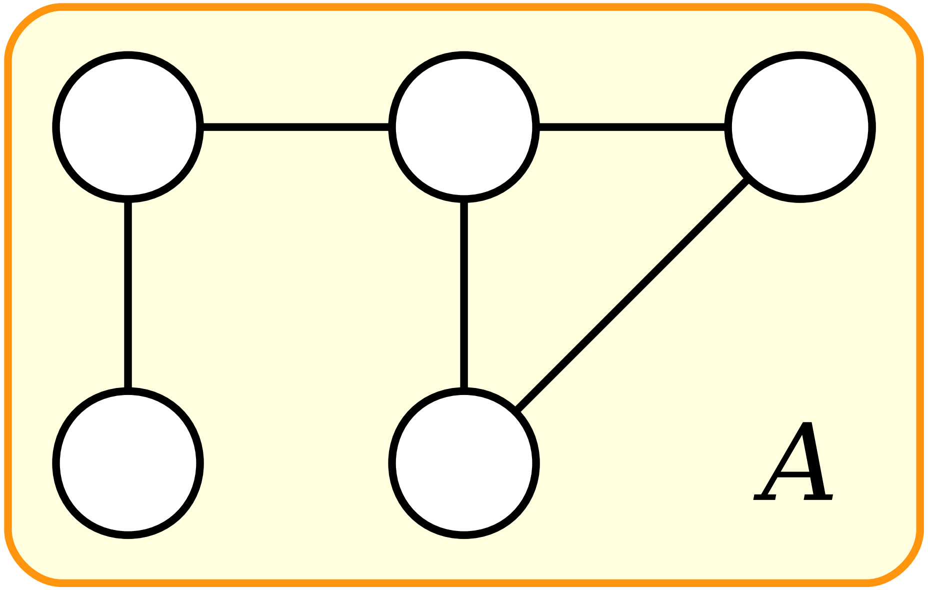


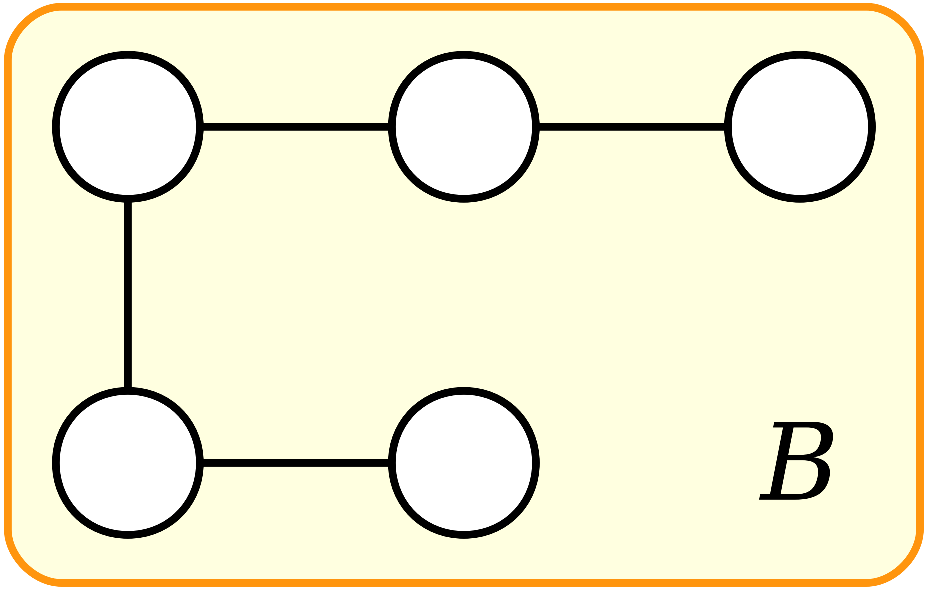


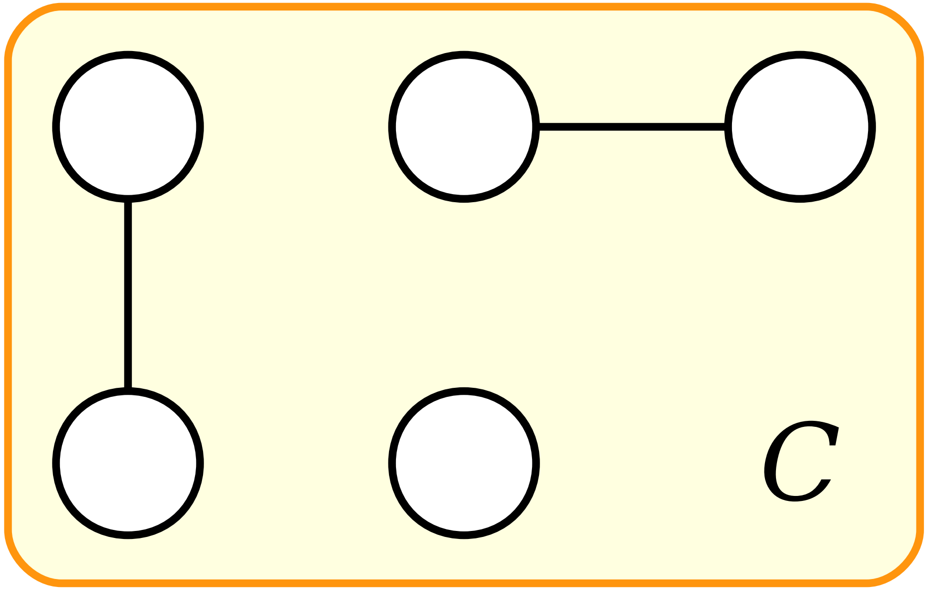


Broadcast Storms

- A ***broadcast storm*** occurs when there's a cycle in the network graph.
- A single message can repeat forever, or exponentially amplify until the network fails.
- ***Solution:*** Don't let the network graph have any cycles.
- A graph $G = (V, E)$ is ***acyclic*** if it has no cycles.







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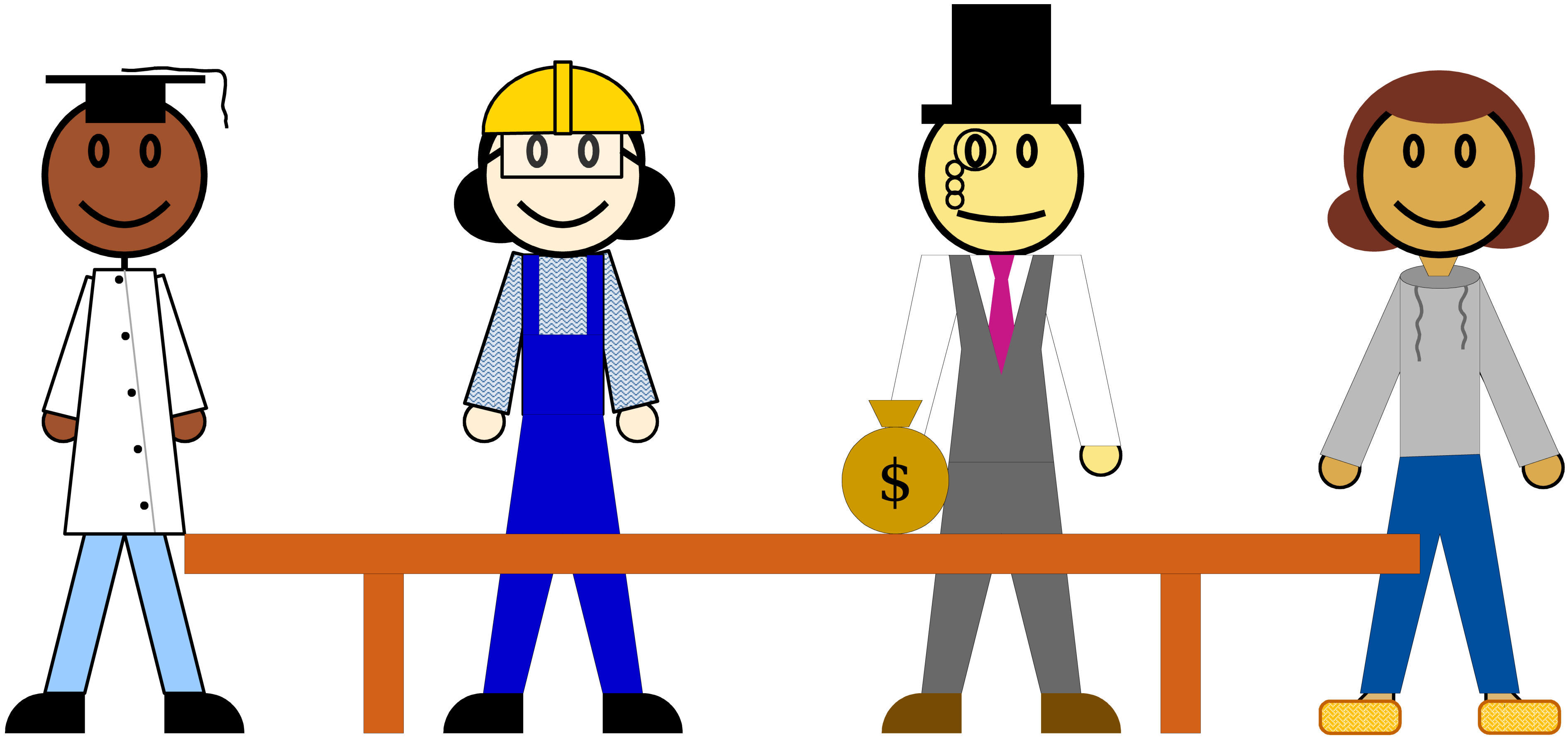
You have a collection of computers that need to be wired up into a LAN. How should you choose the shape of the network?



CTO

Connected,
No Cycles

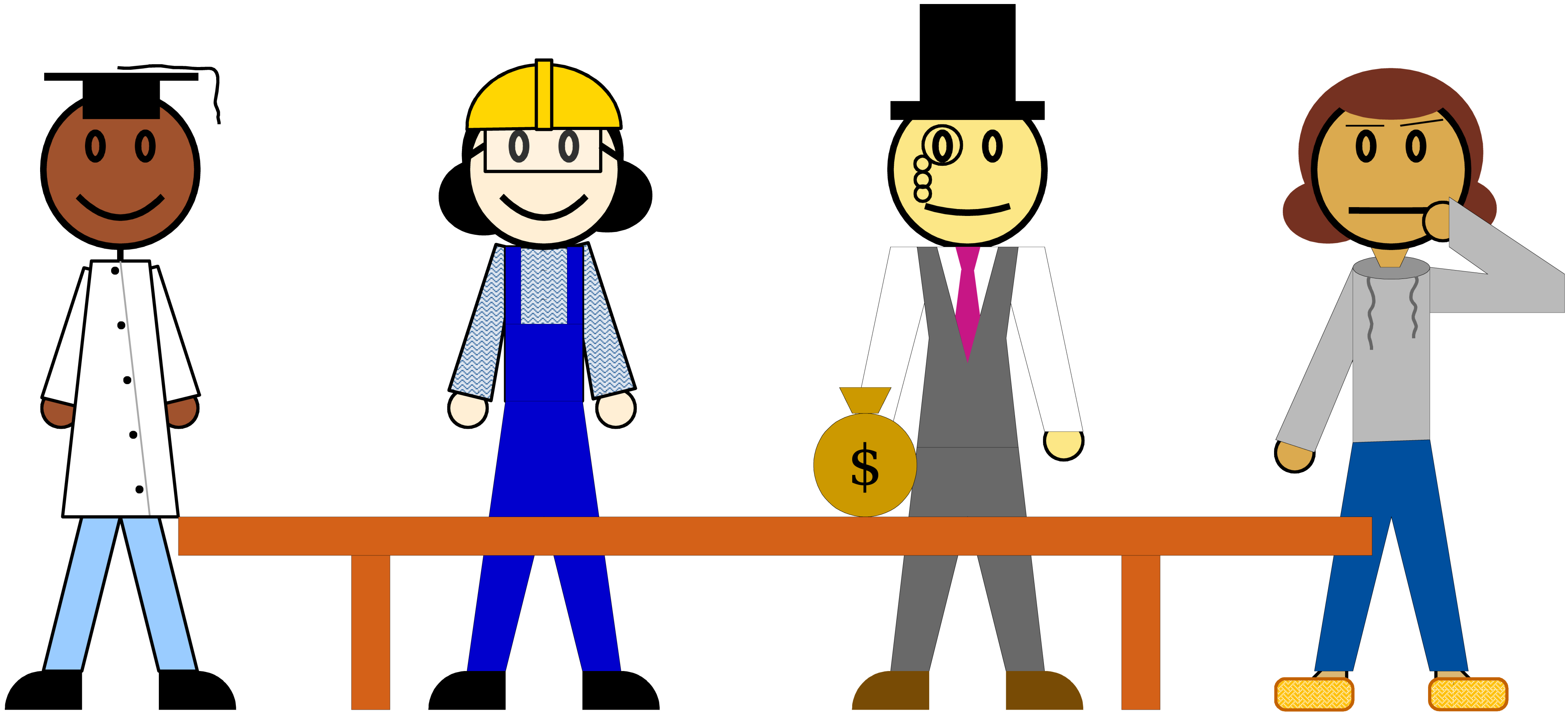
COO

Most Links,
No Cycles

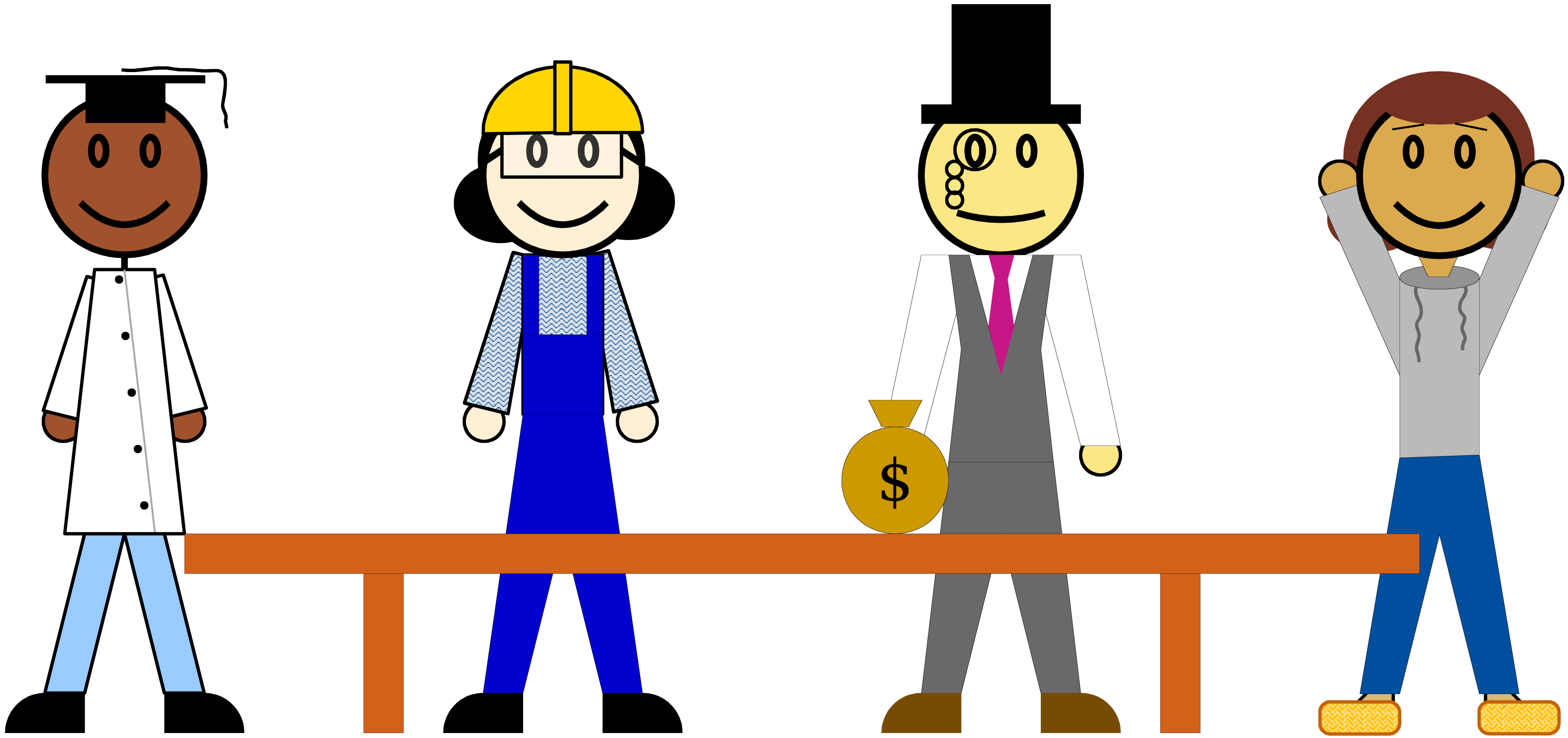
CFO

Fewest Links,
Connected

CEO



CTO	***COO***	***CFO***	***CEO***
Connected, No Cycles	Most Links, No Cycles	Fewest Links, Connected	



CTO

Connected,
No Cycles

COO

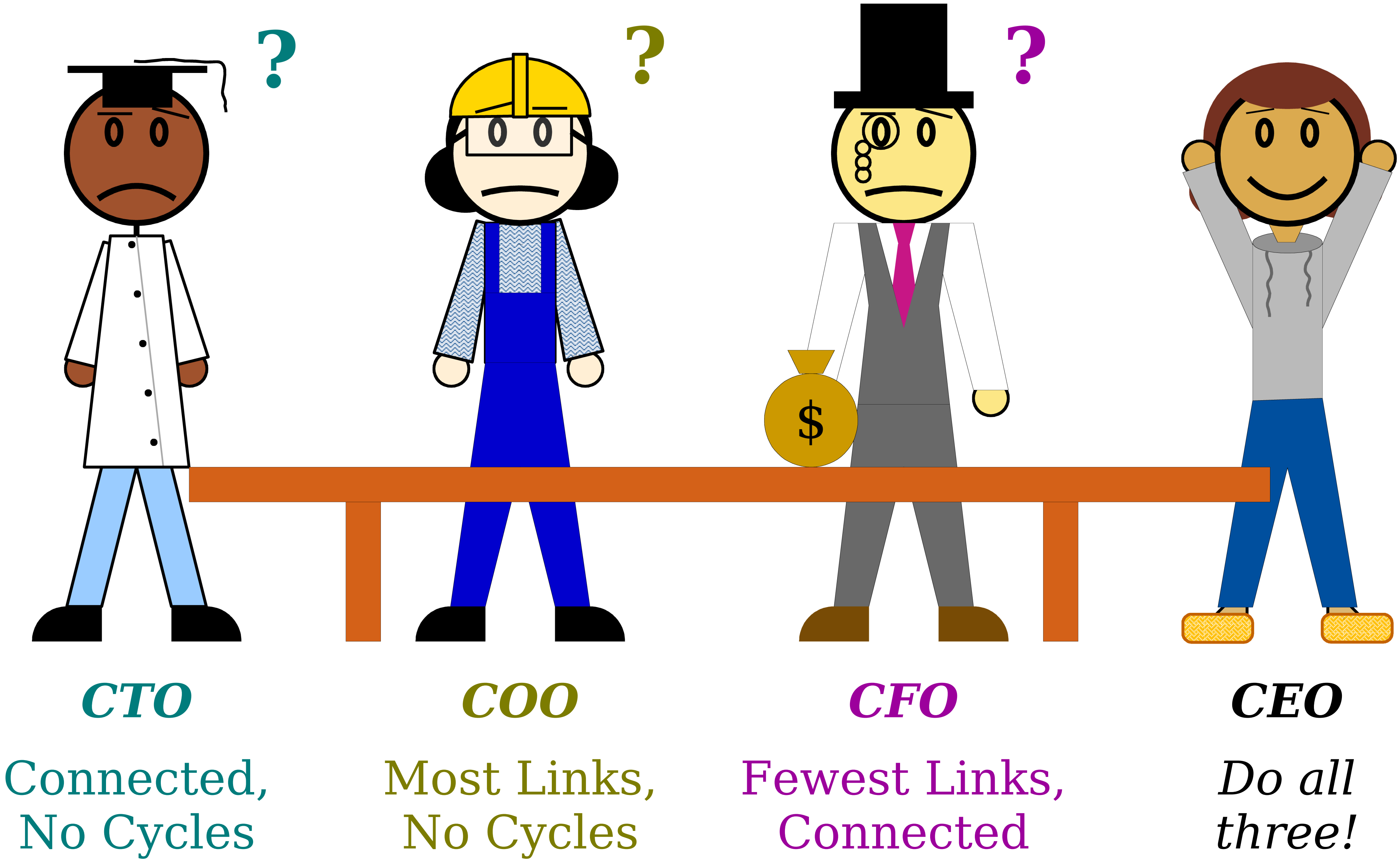
Most Links,
No Cycles

CFO

Fewest Links,
Connected

CEO

Do all three!


?
?
?
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Connected
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Do all
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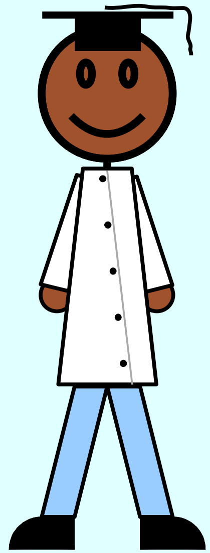
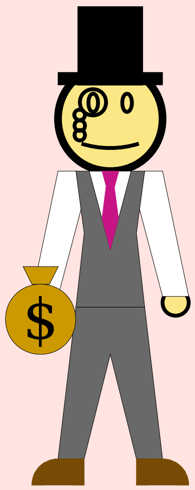


Connected, Acyclic

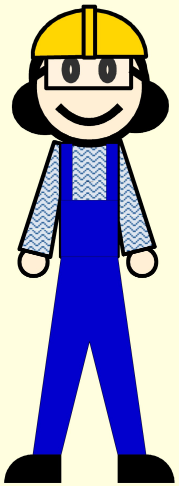


Minimally Connected

(Connected, but deleting any edge disconnects its endpoints.)

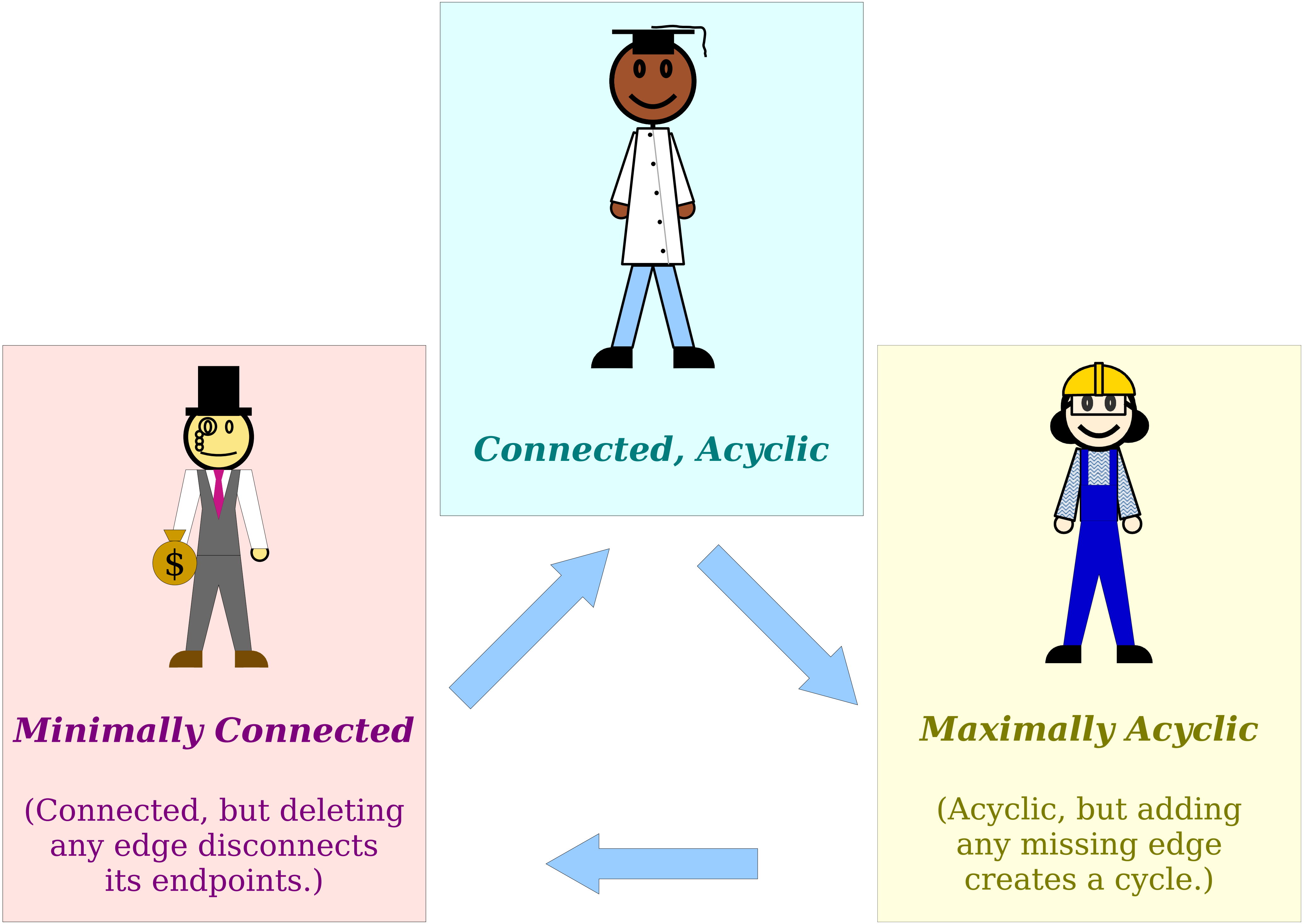
If *any* of these conditions hold, then *all* of these conditions hold.

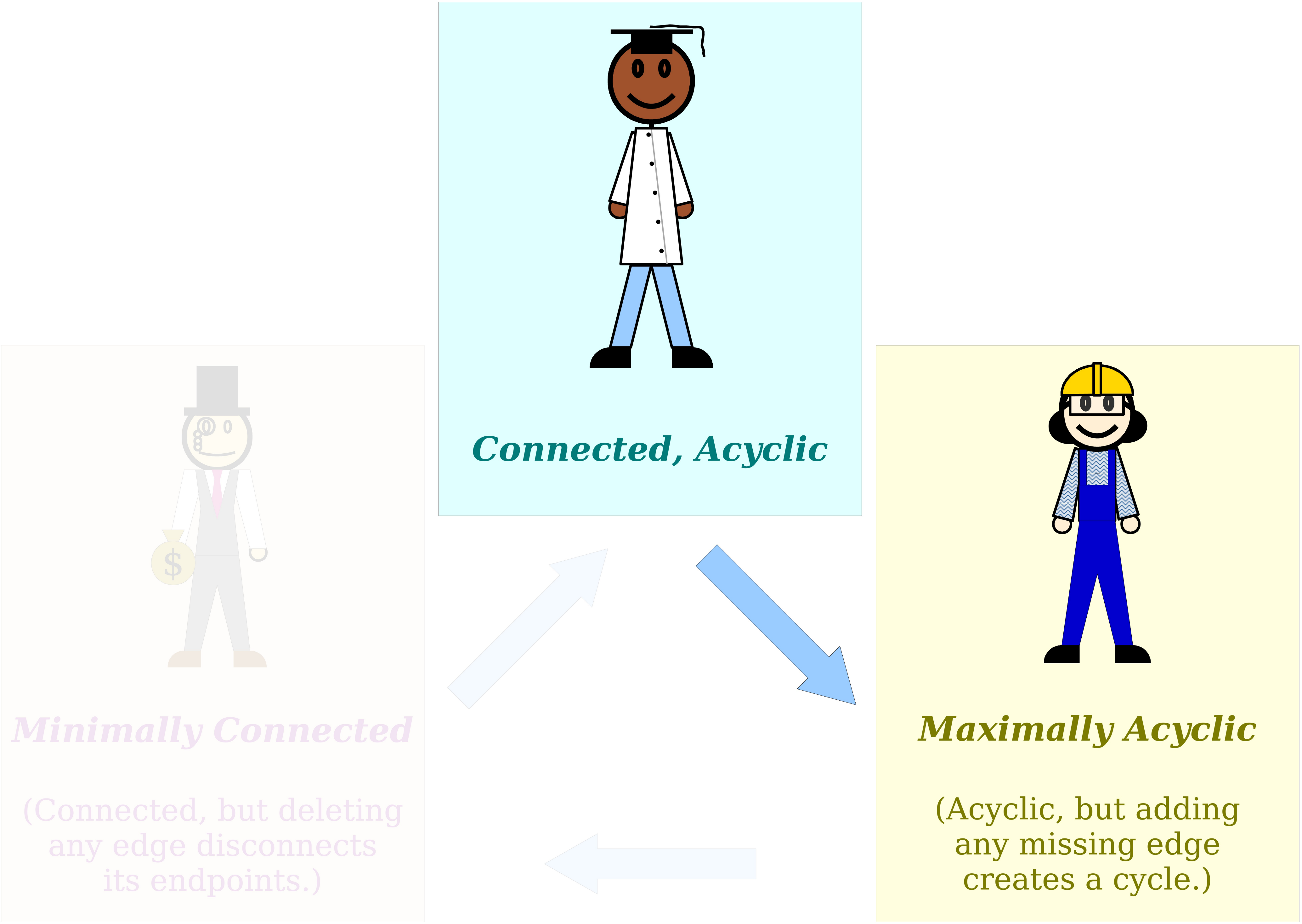
A graph with any of these properties is called a ***tree***.



Maximally Acyclic

(Acyclic, but adding any missing edge creates a cycle.)


Connected, Acyclic
Minimally Connected
(Connected, but deleting any edge disconnects its endpoints.)
Maximally Acyclic
(Acyclic, but adding any missing edge creates a cycle.)


Connected, Acyclic
Minimally Connected
(Connected, but deleting any edge disconnects its endpoints.)
Maximally Acyclic
(Acyclic, but adding any missing edge creates a cycle.)

Trees

- ***Theorem:*** Let $T = (V, E)$ be a graph. The following are equivalent:
  - T is connected and acyclic. (CTO perspective.)
  - T is ***maximally acyclic***: T has no cycles, and adding any missing edge $\{x, y\}$ creates a cycle. (COO perspective.)
  - T is ***minimally connected***: T is connected, and deleting any edge $\{x, y\}$ from T disconnects x from y. (CFO perspective.)
- A graph meeting any of these three sets of requirements is called a ***tree***.

Theorem: Let $T = (V, E)$ be a graph. If T is connected and acyclic, then T is maximally acyclic.

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Proof:

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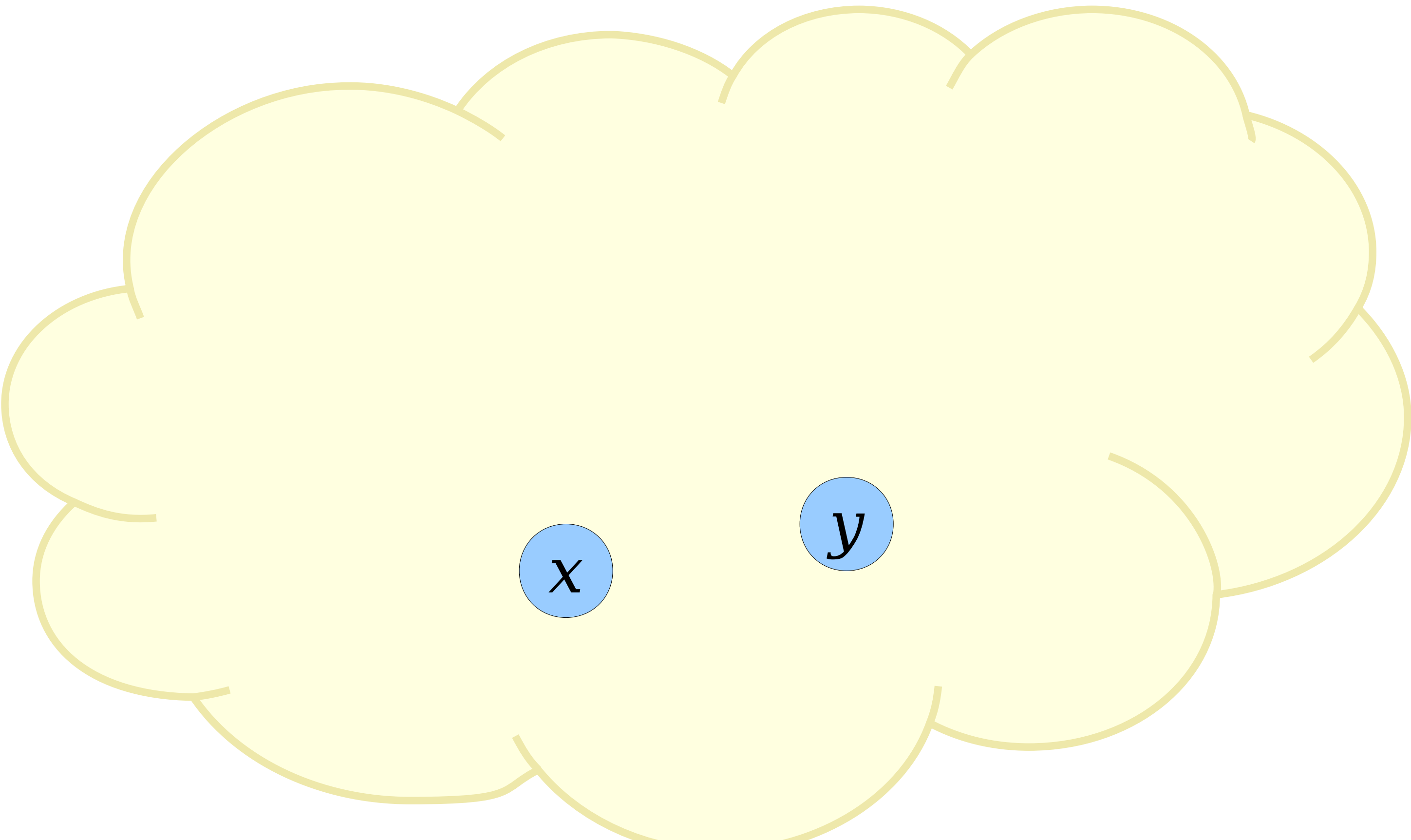

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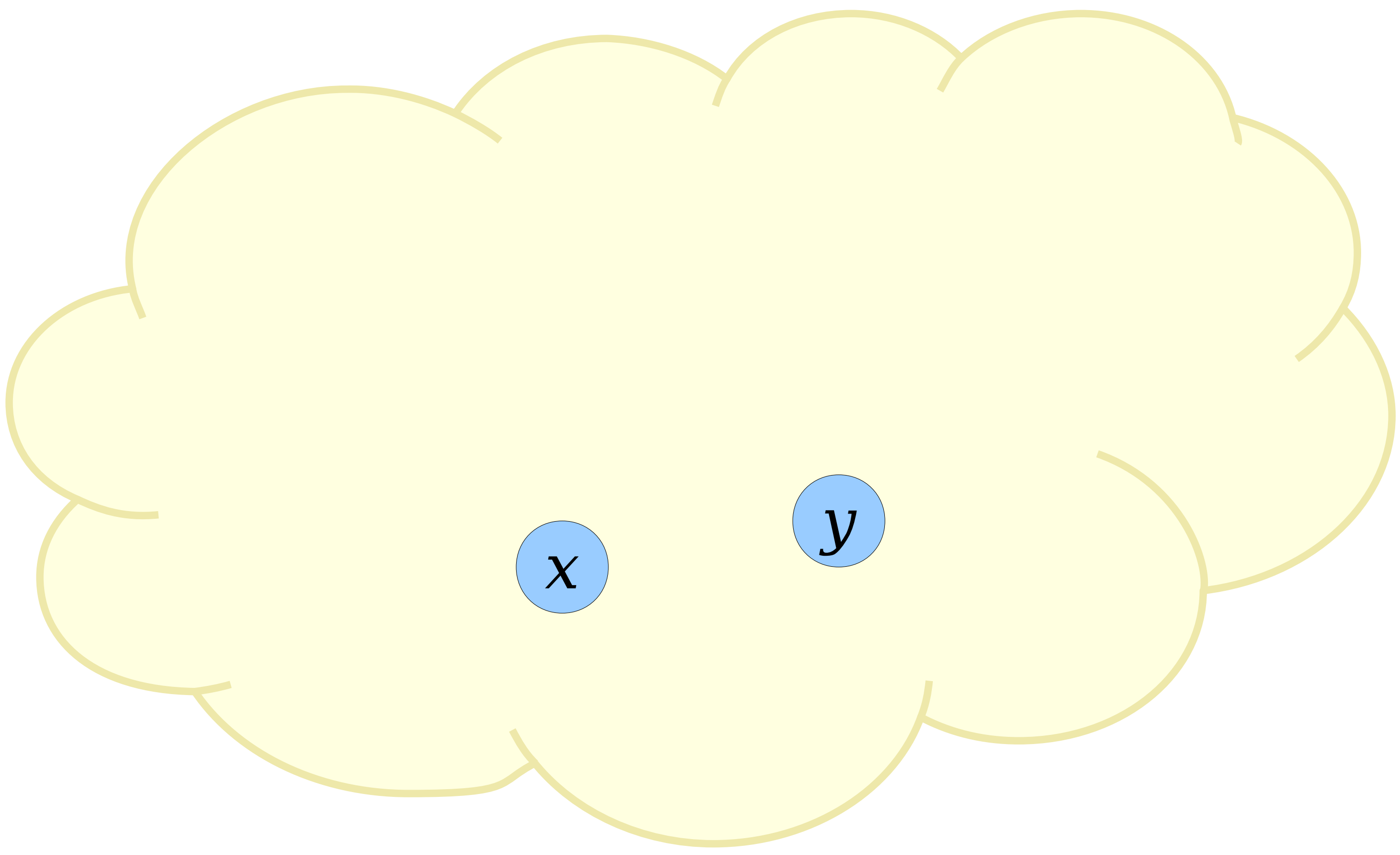

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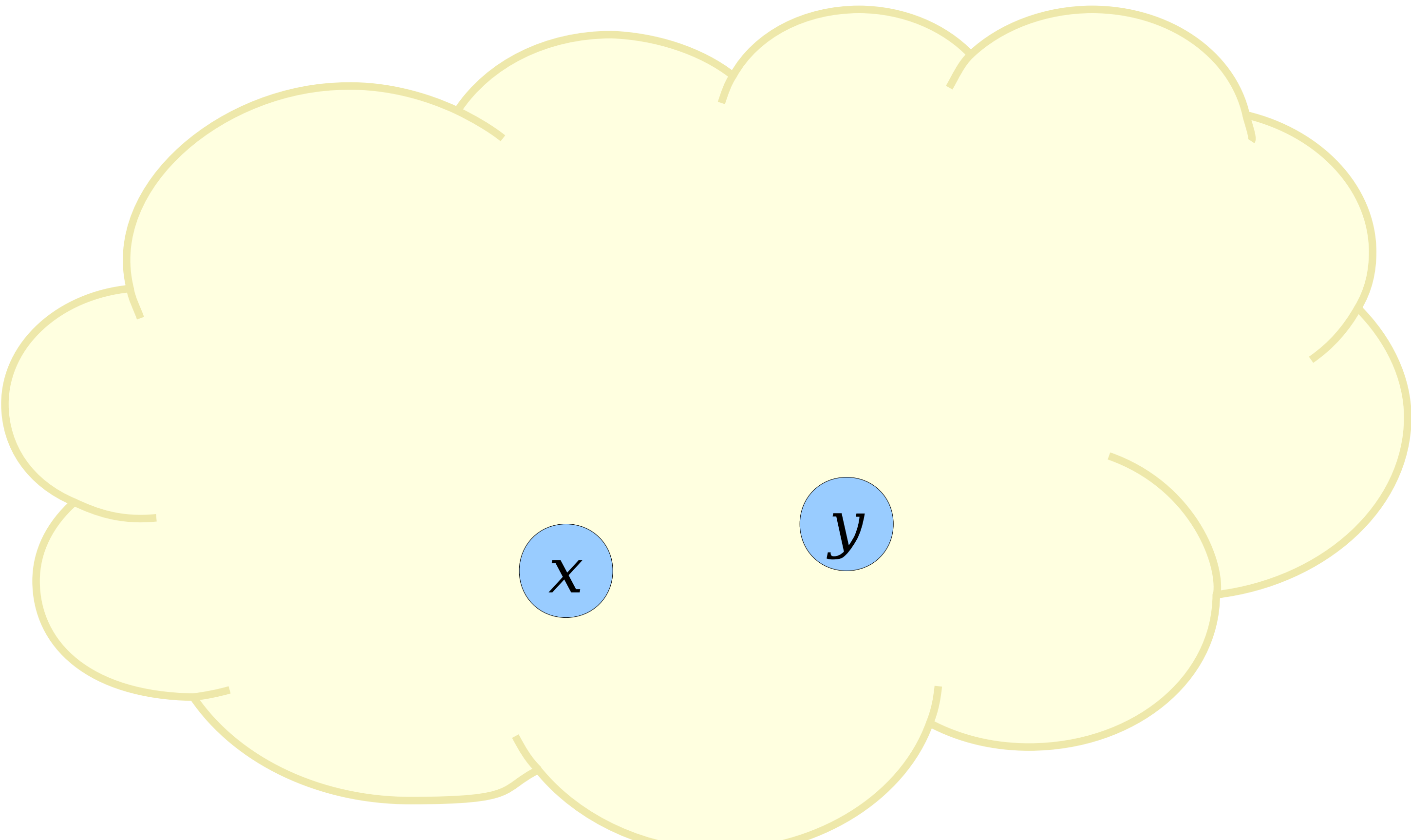

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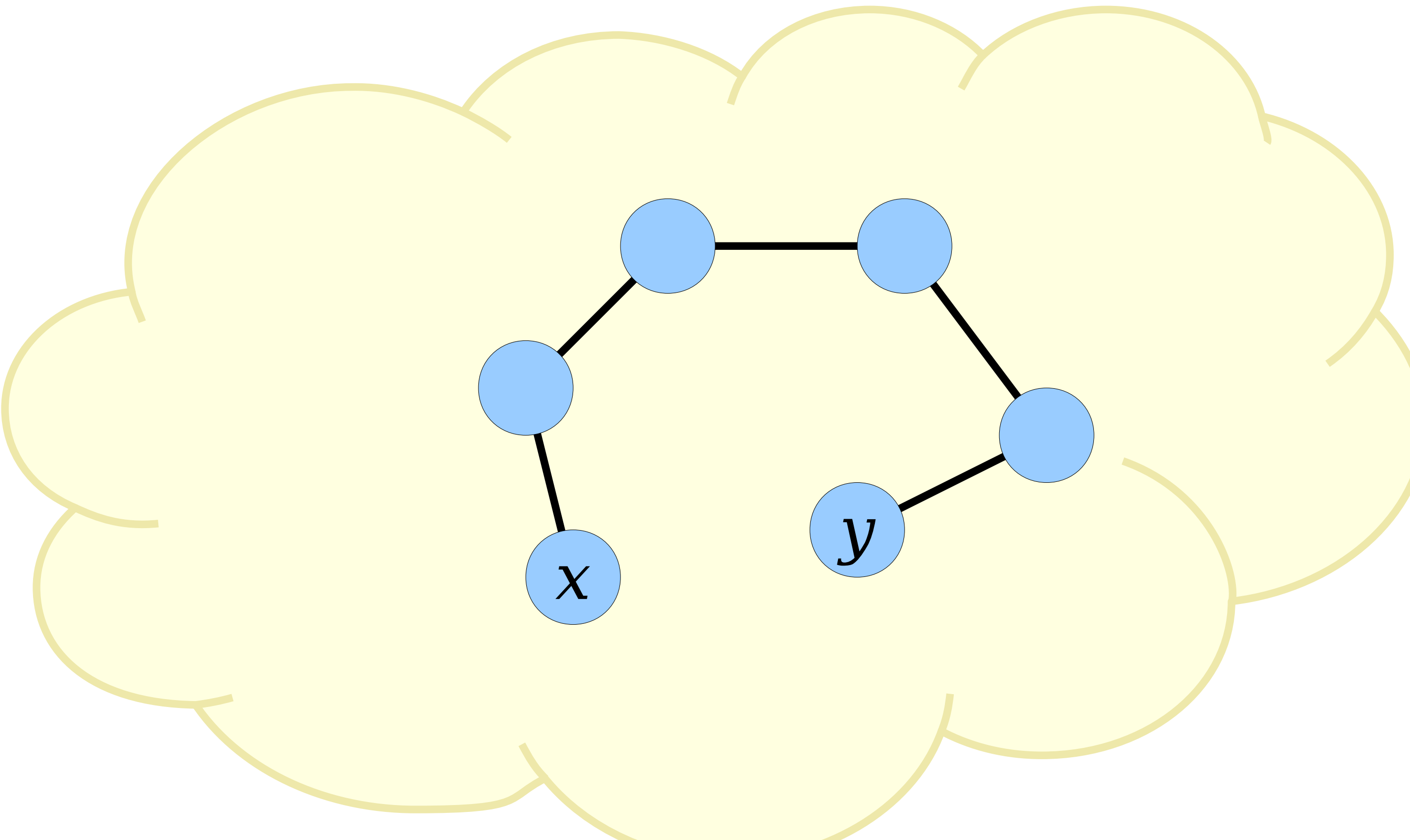

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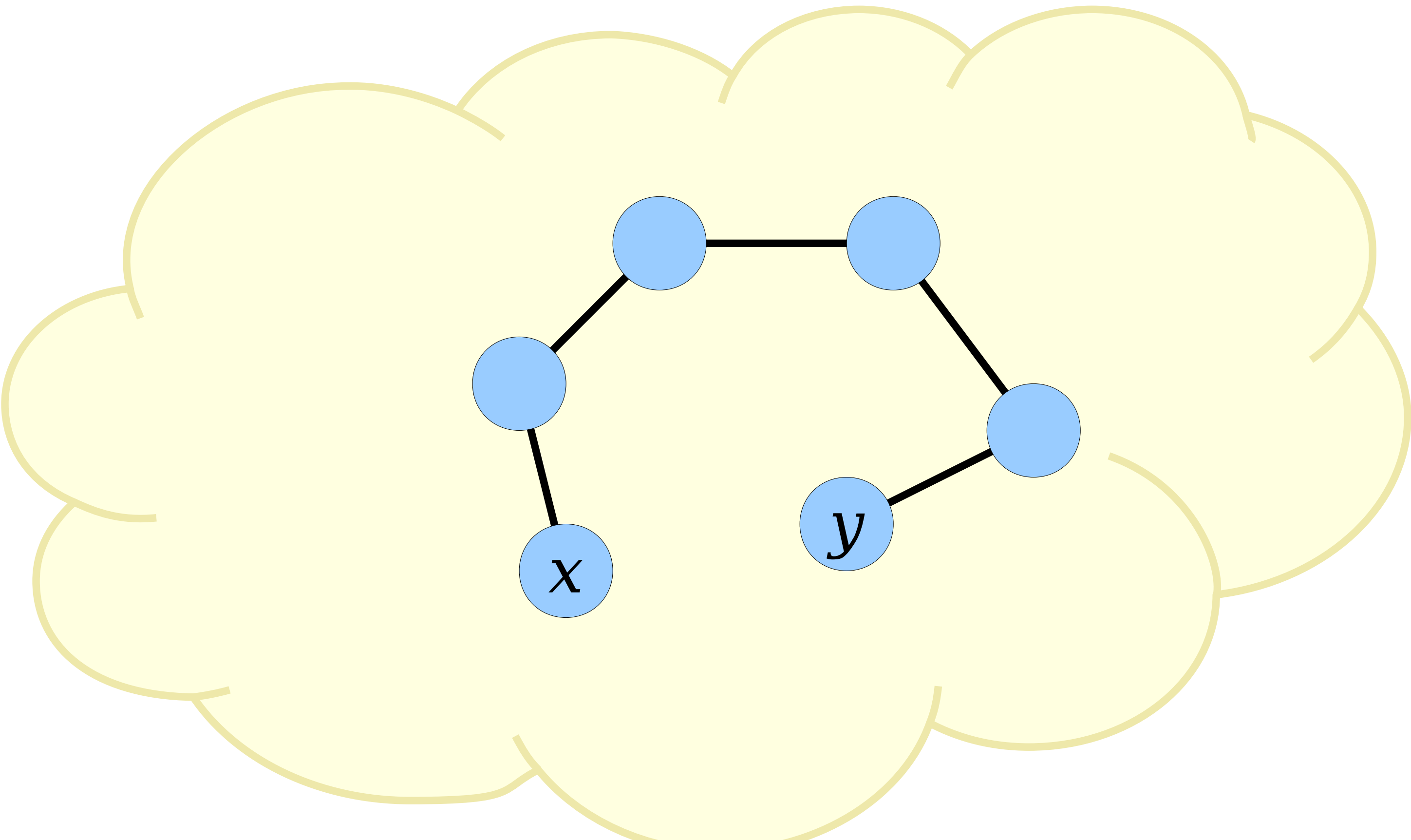

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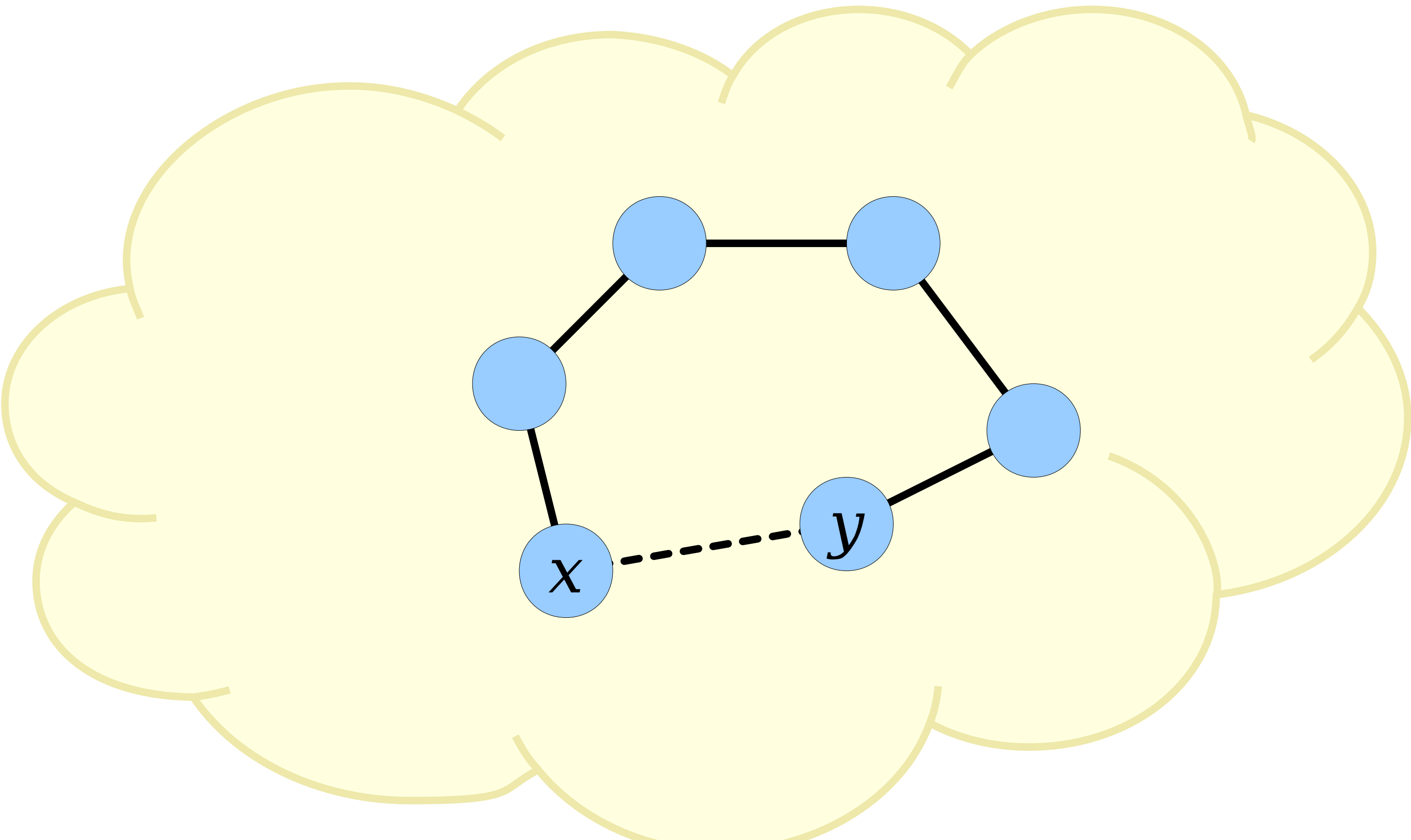

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Check the appendix for the other two steps of the proof.

More to Explore

- A tree kind of seems like a bad way to design a network. (Why?)
- Actual local area networks allow for cycles. They use something called the ***spanning tree protocol*** (***STP***) to selectively disable links to form a tree.
- Routing through the full internet – not just within a LAN – is a fascinating topic in its own right.
- Take CS144 (networking) for details!

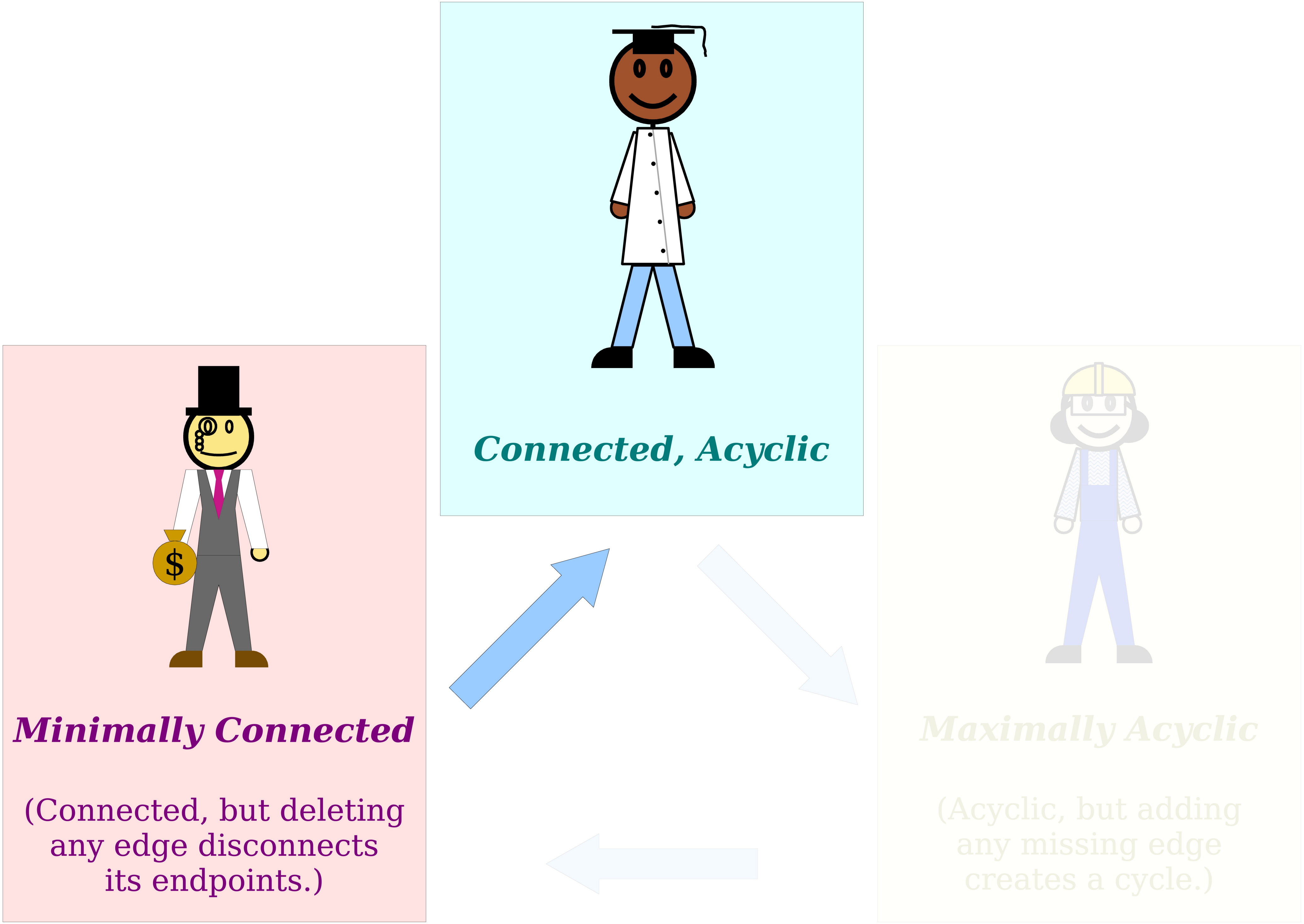
Recap from Today

- ***Walks*** and ***closed walks*** represent ways of moving around a graph. ***Paths*** and ***cycles*** are "redundancy-free" walks and cycles.
- ***Trees*** are graphs that are connected and acyclic. They're also minimally-connected graphs and maximally-acyclic graphs.
- Trees have applications throughout CS, including networking.

Next Time

- ***The Pigeonhole Principle***
  - A simple, powerful, versatile theorem.
- ***Graph Theory Party Tricks***
  - Applying math to graphs of people!
- ***A Little Movie Puzzle***
  - Who watched what?

Appendix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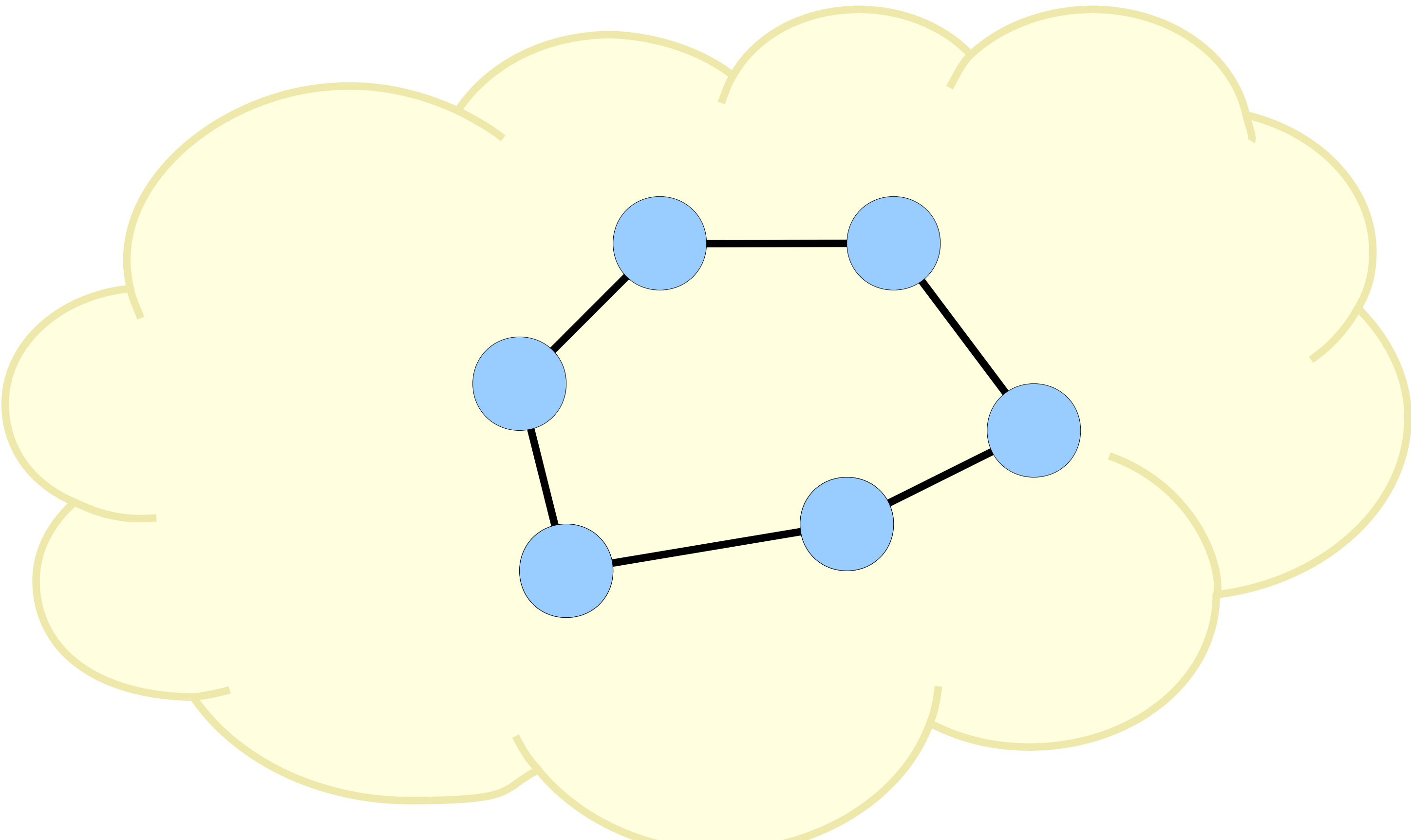

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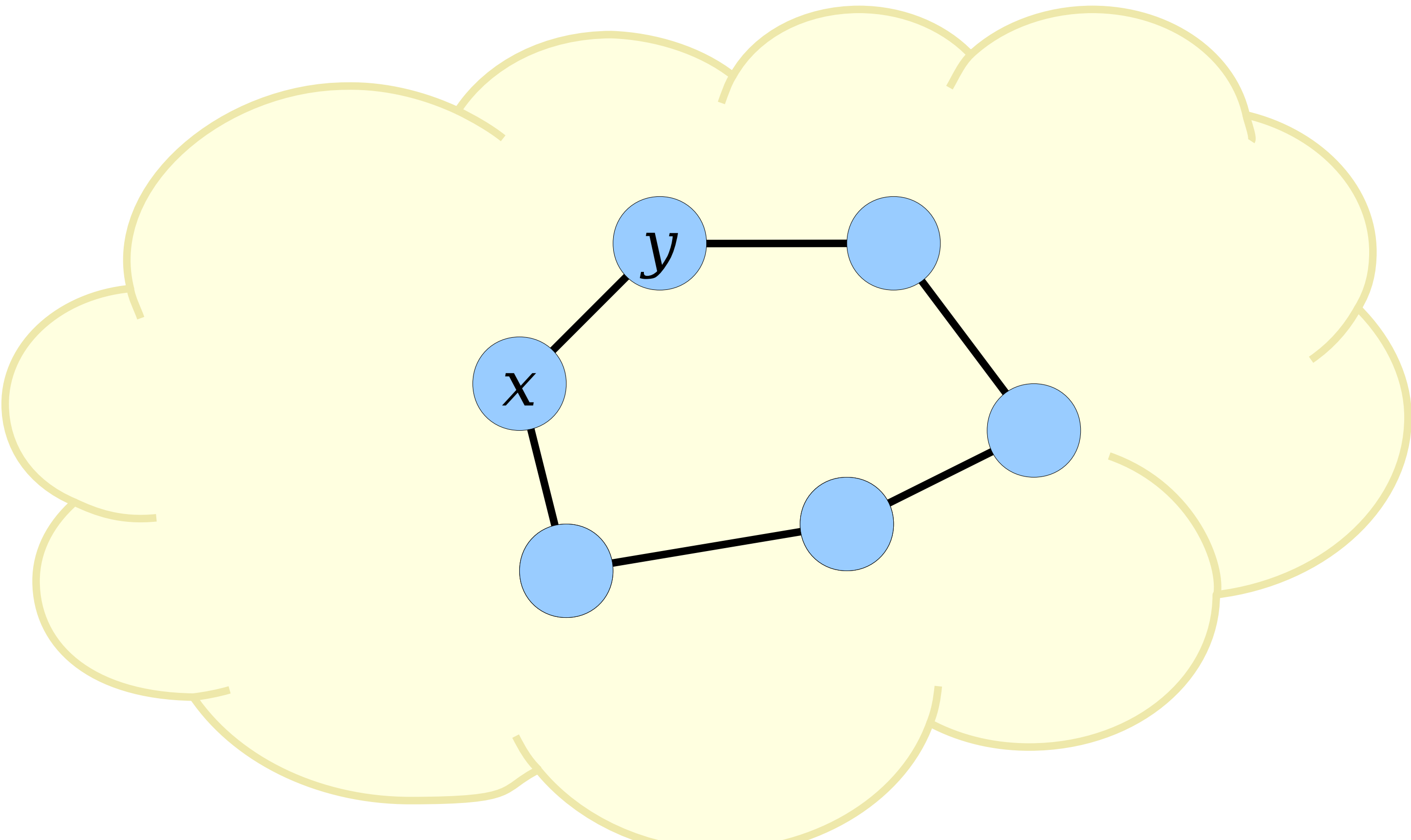

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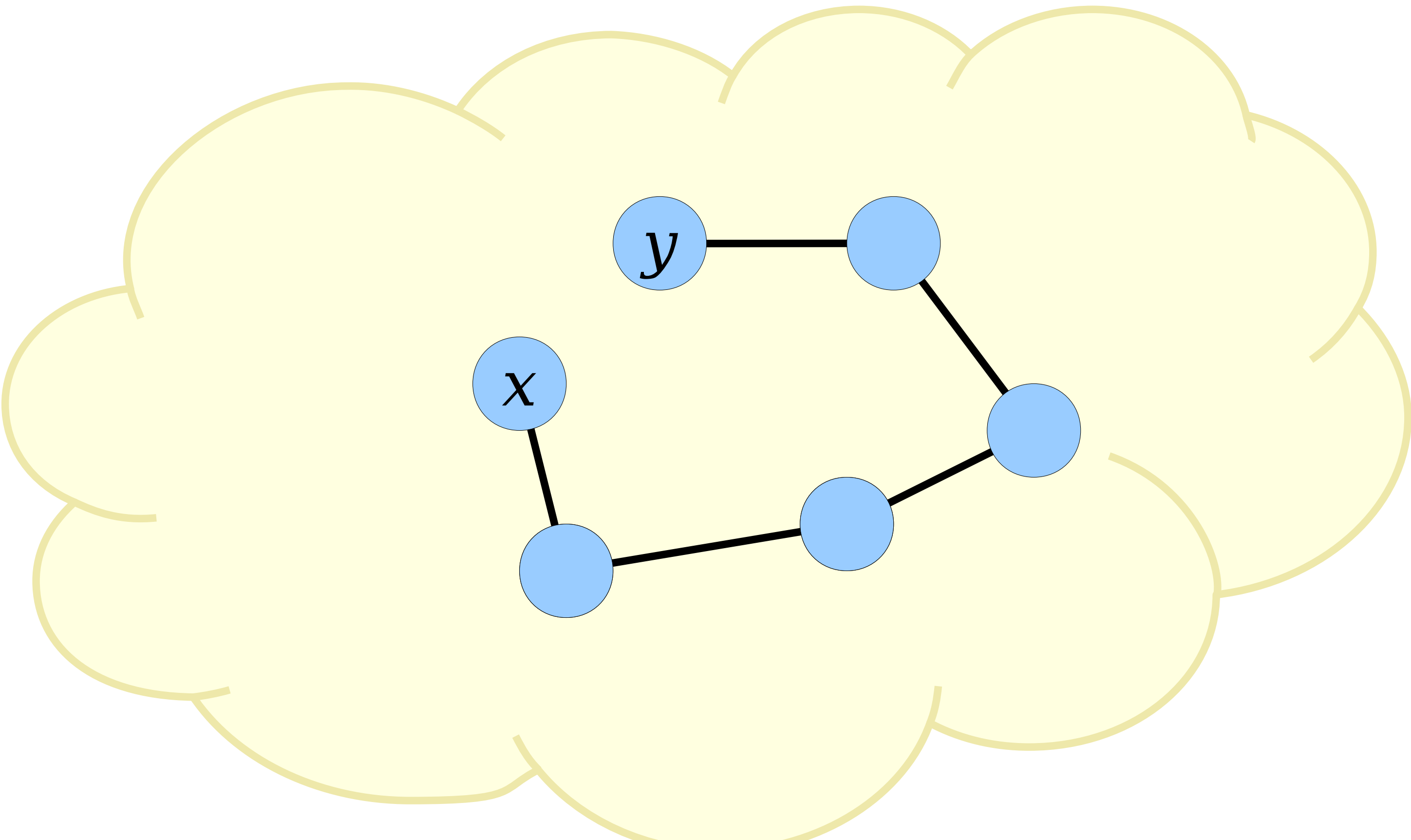

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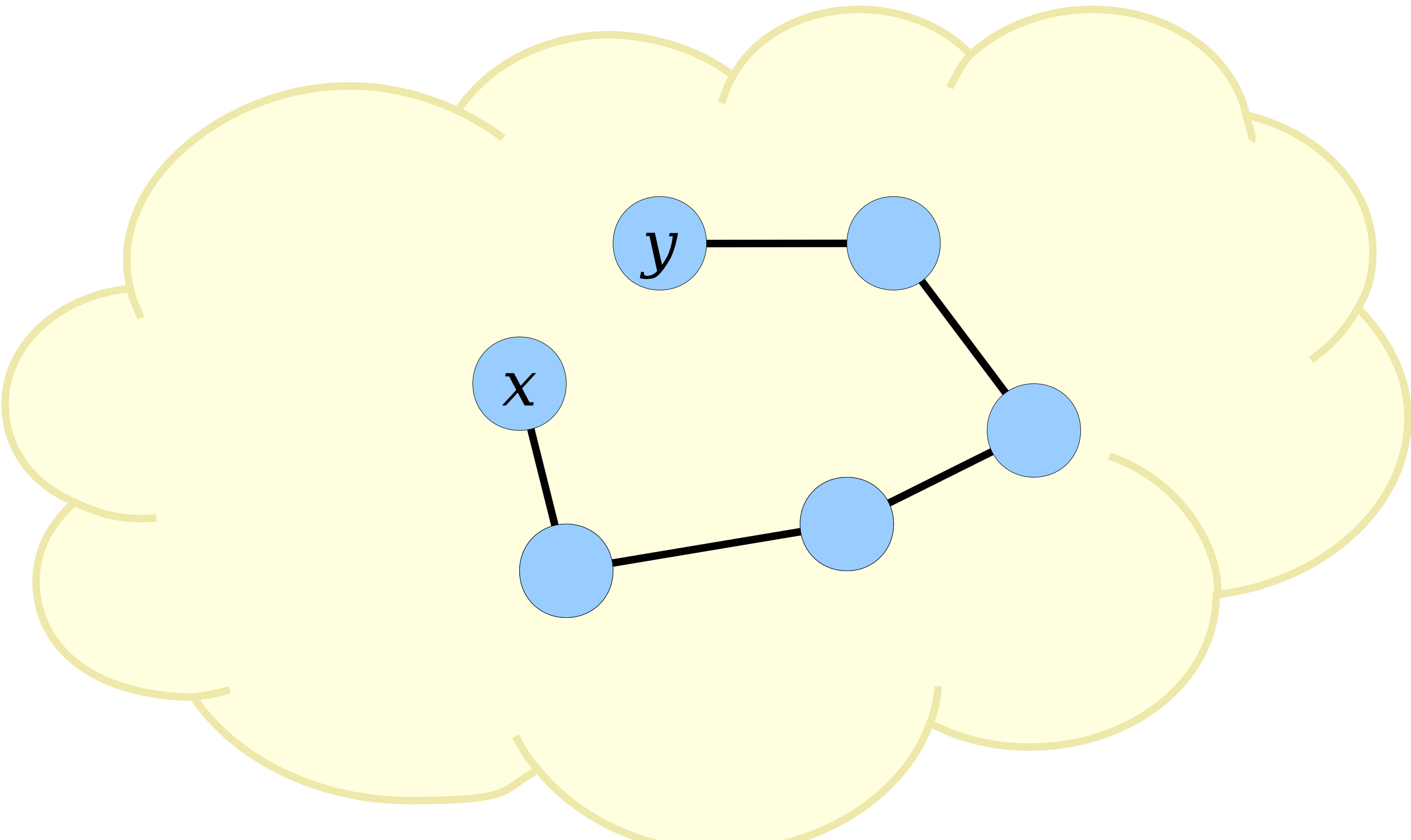

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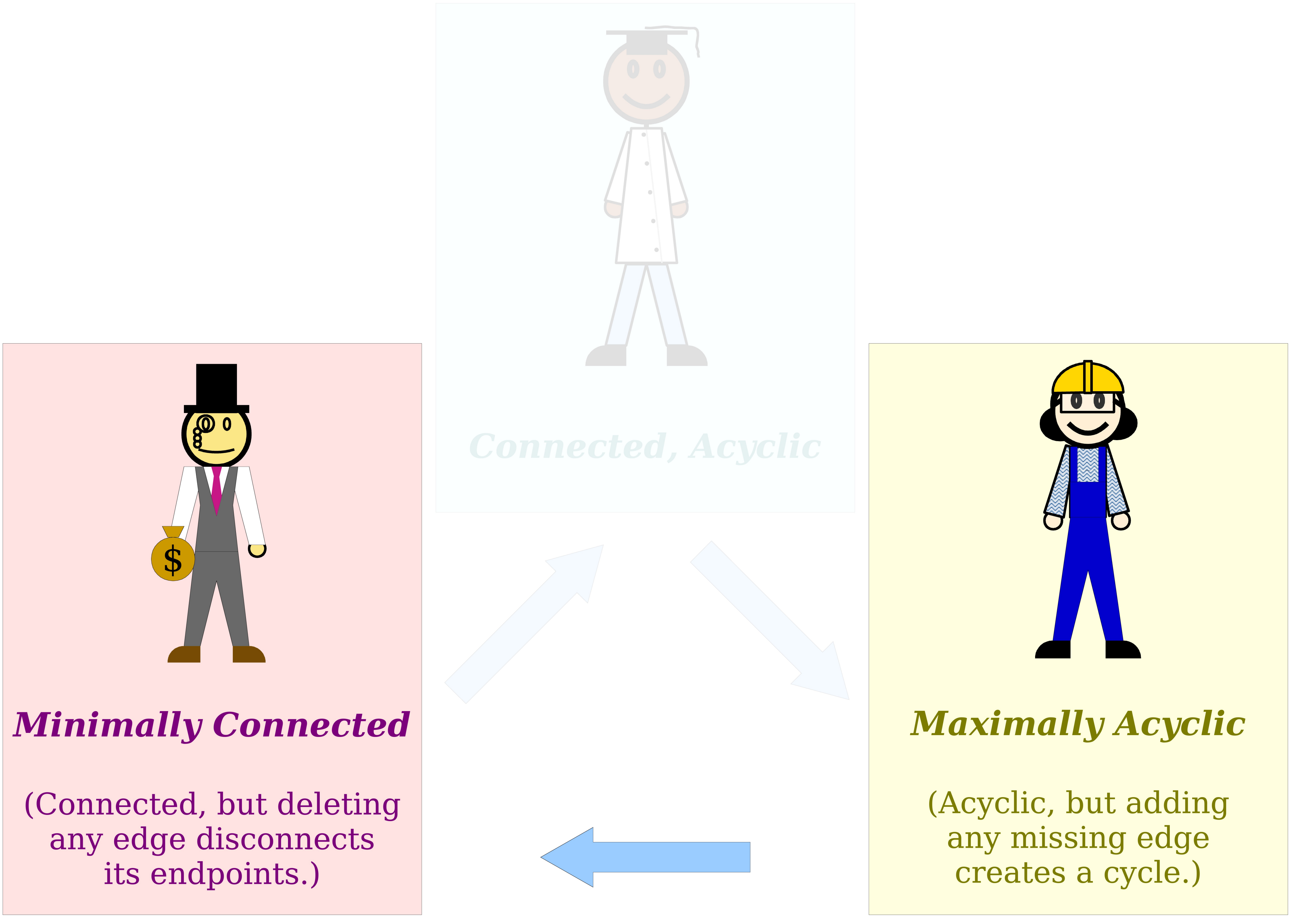
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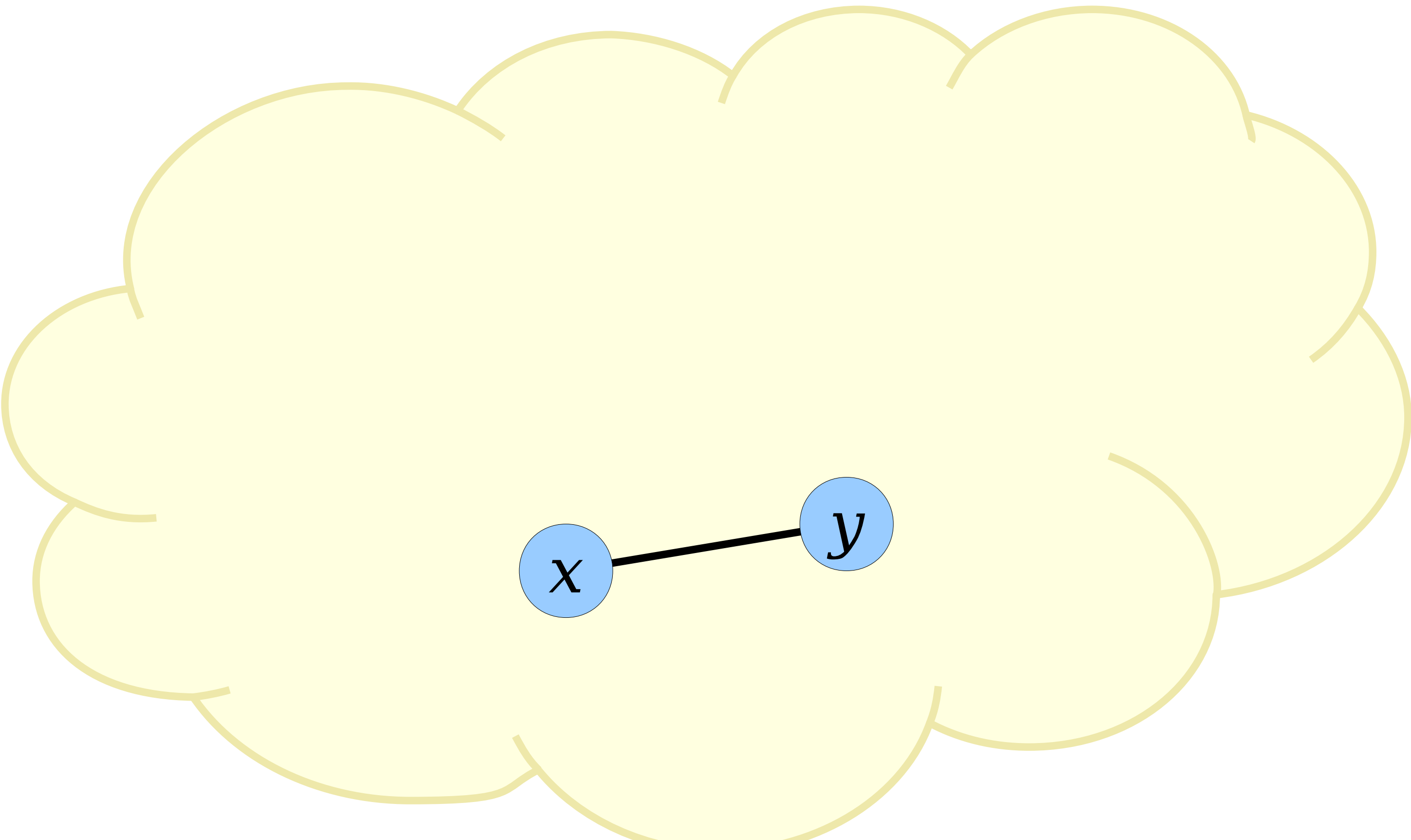

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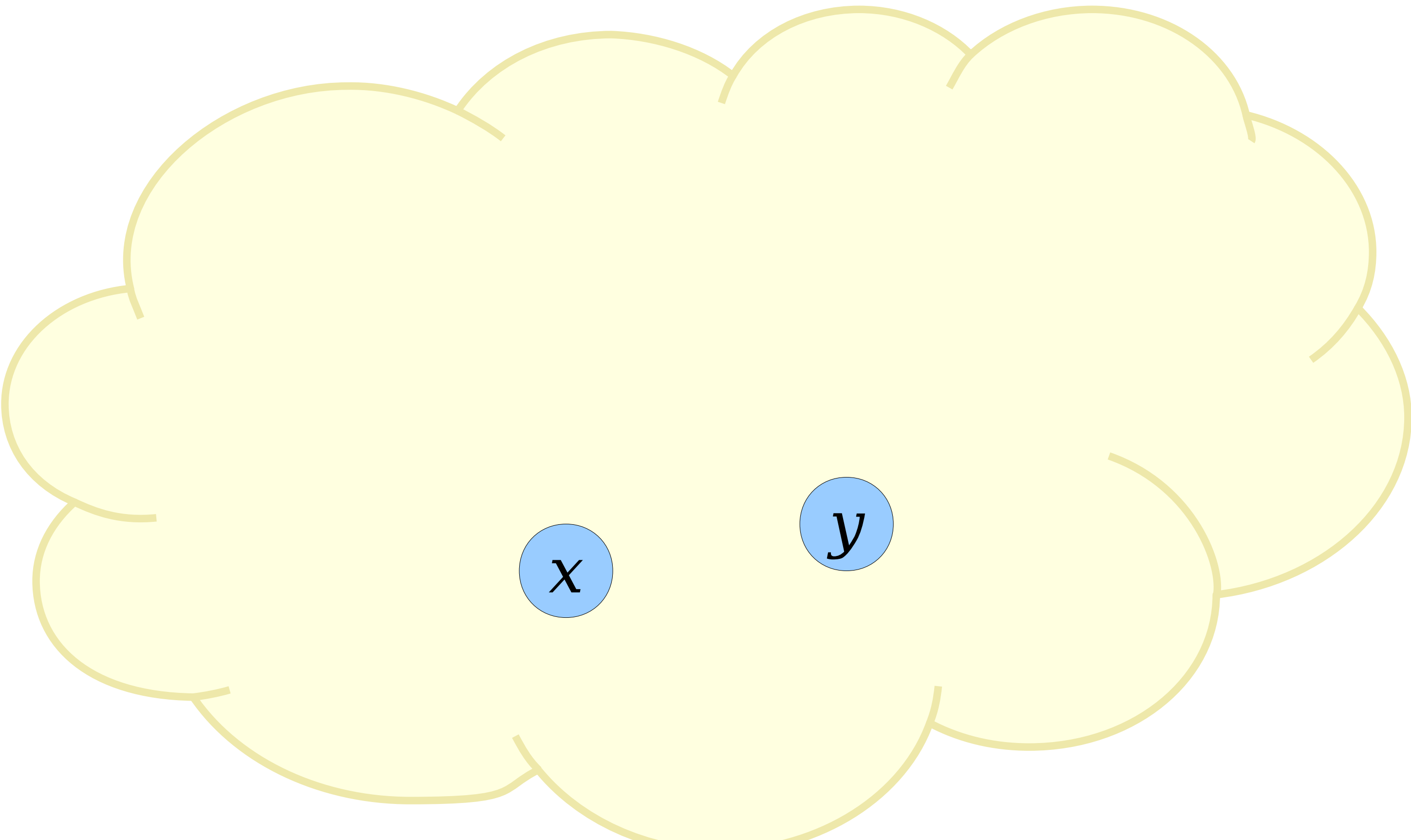

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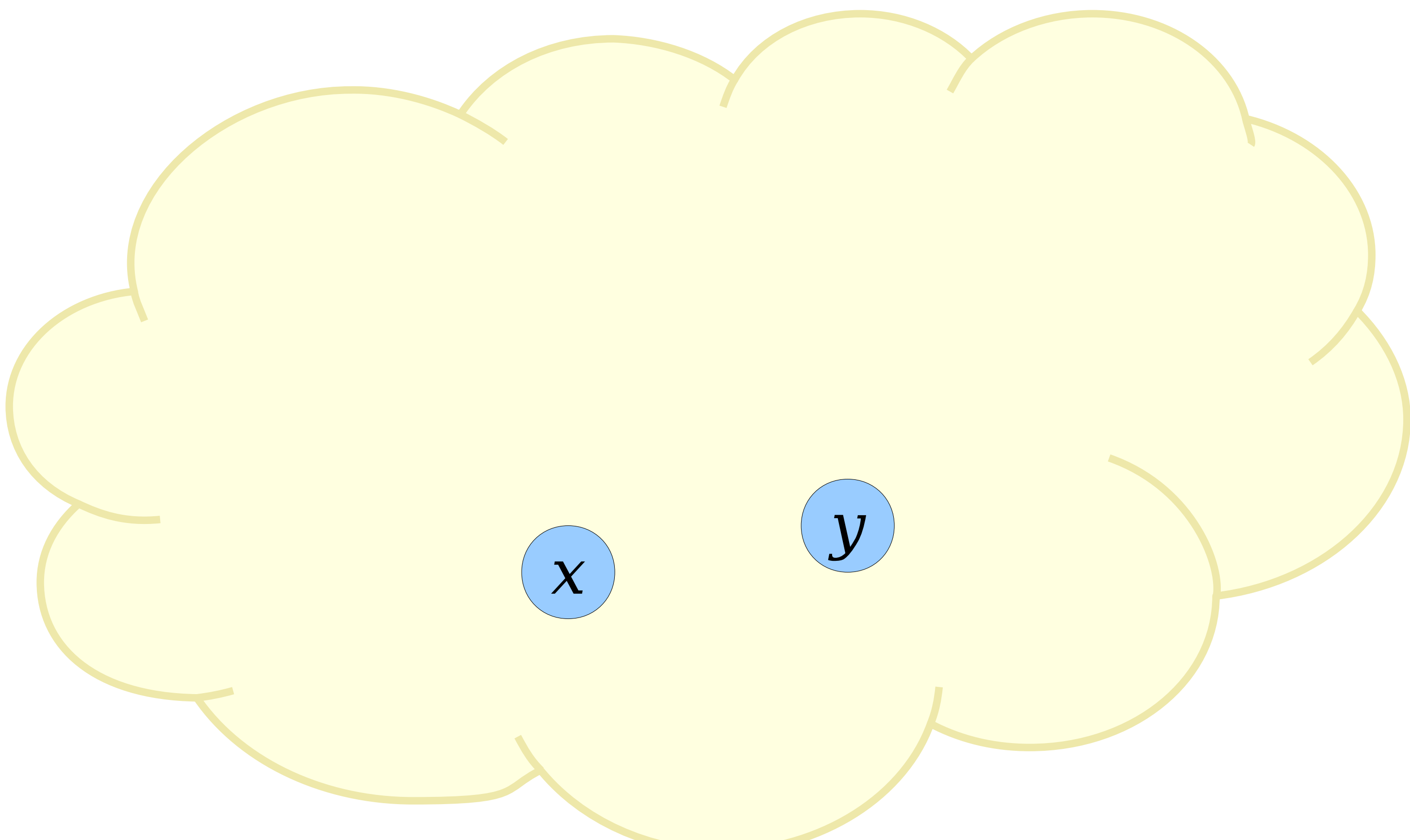

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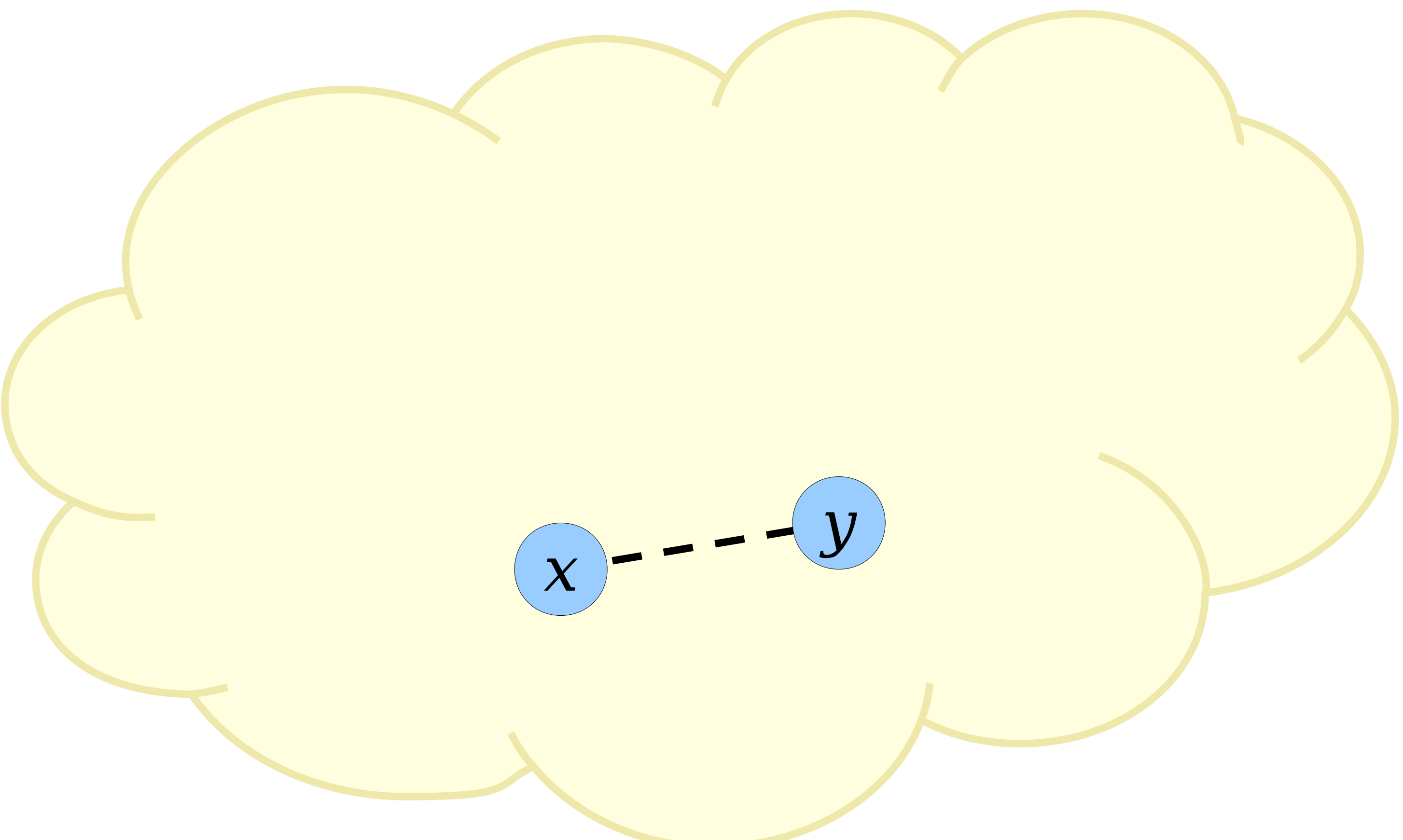

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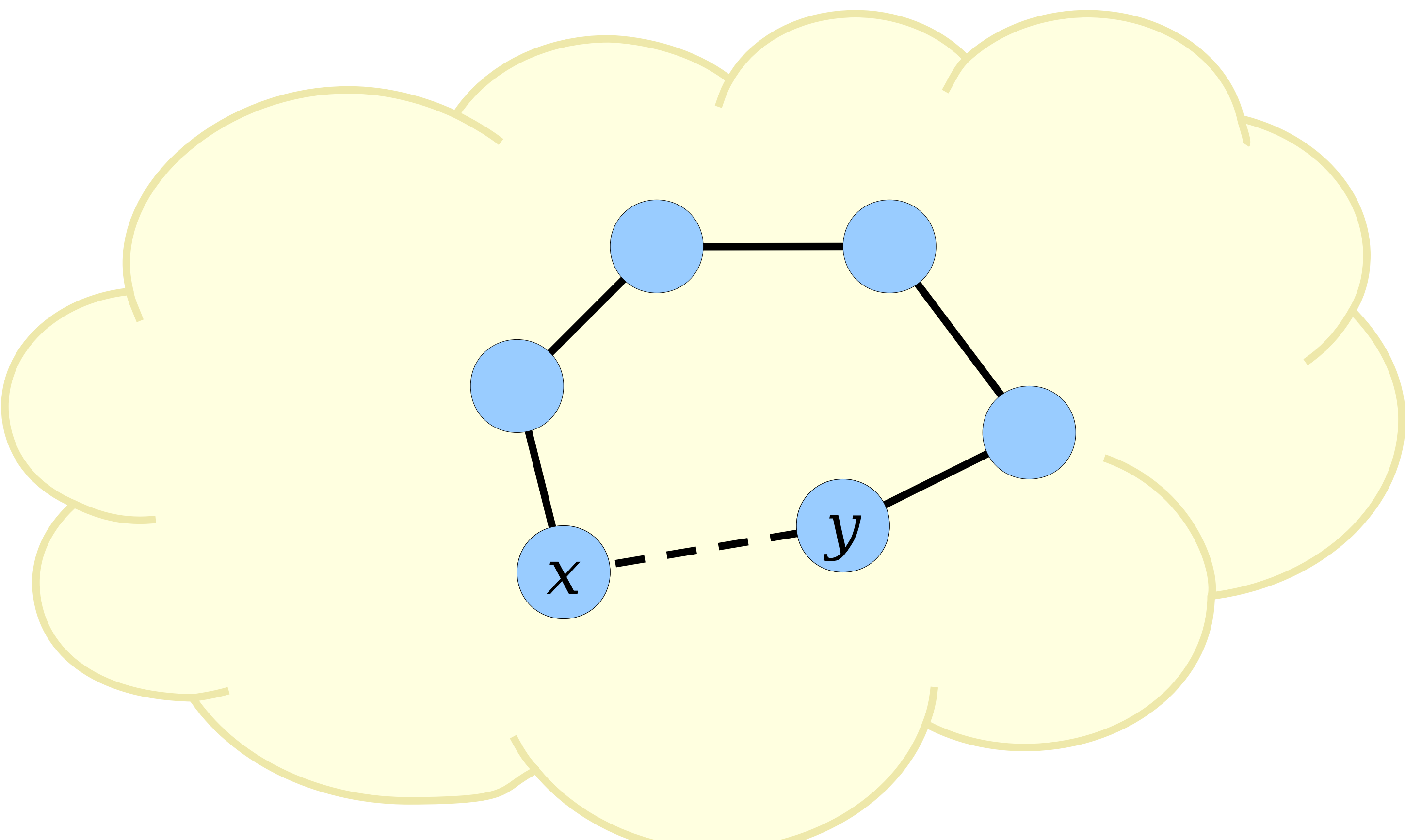

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Case 1: $\{x, y\} \in E$.

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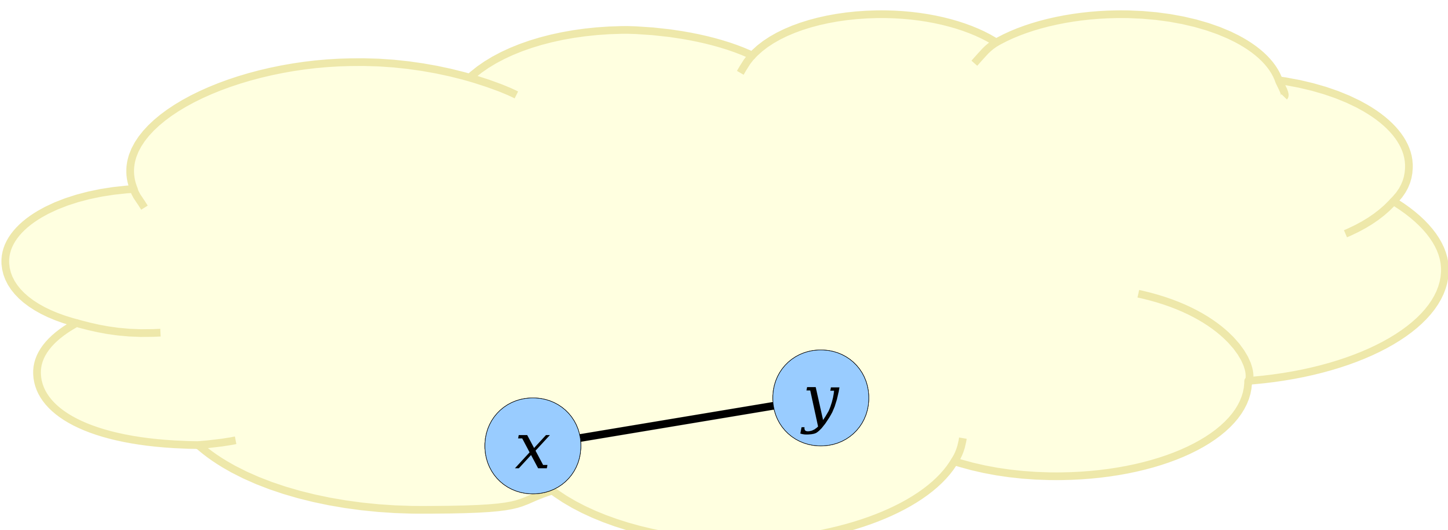

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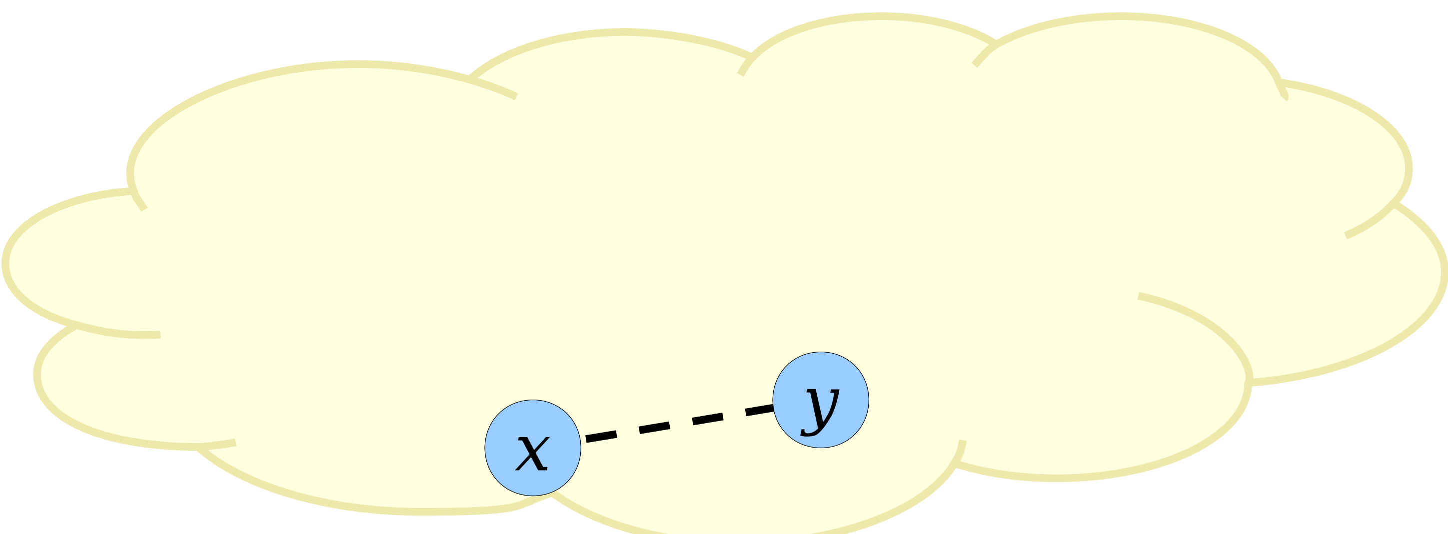

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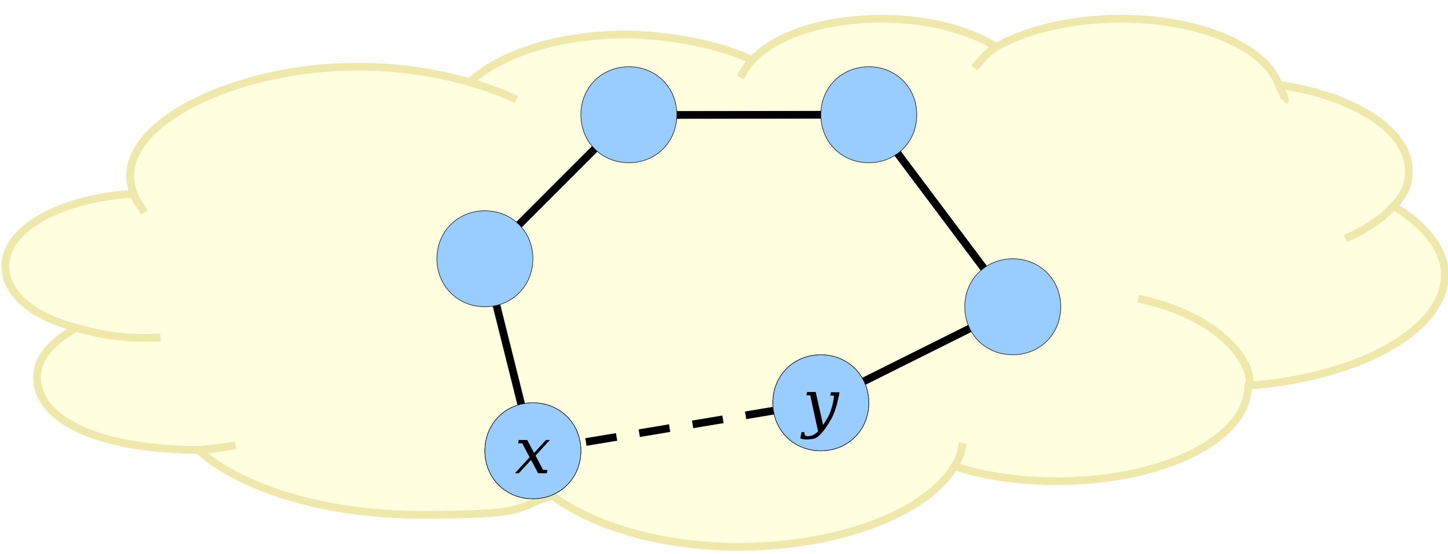

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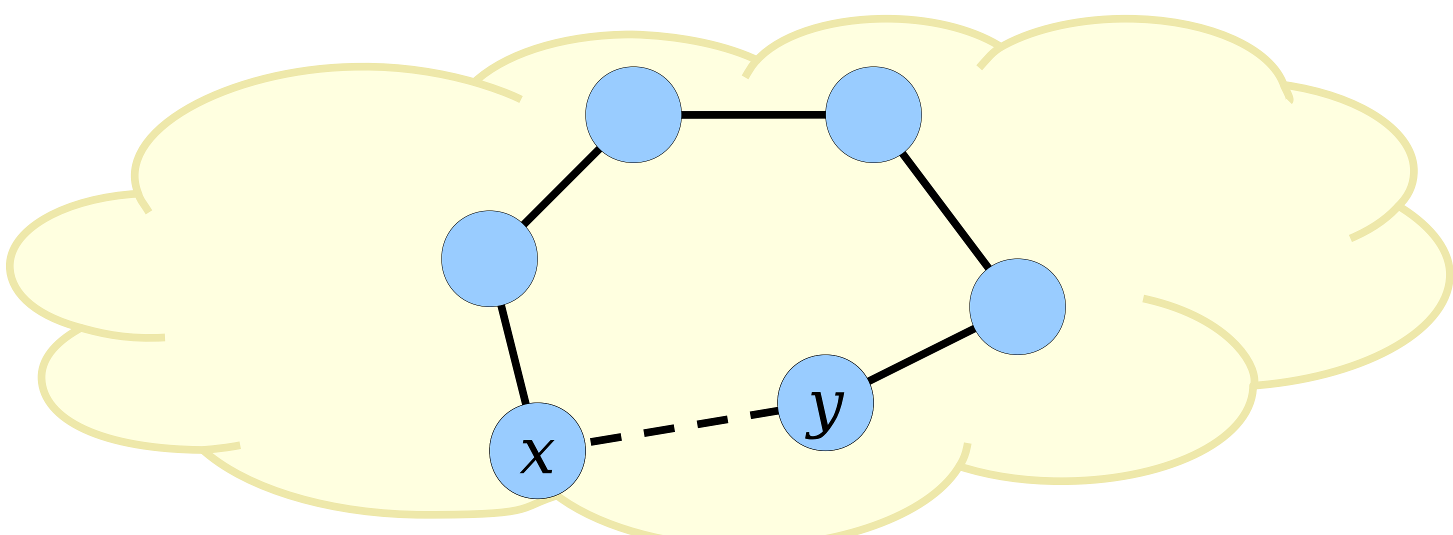

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